ME 5350 - Homework 3

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Part I

Problem 3.9

Problem Statement

Does equilibrium exist for the following stress distribution in the absence of body force?

$$\sigma_x = 3x^2 + 4xy - 8y^2$$
$$\tau_{xy} = \frac{1}{2}x^2 - 6xy - 2y^2$$
$$\sigma_y = 2x^2 + xy + 3y^2$$
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Solution

This problem can be solved by inserting the stress components into the equilibrium equation. The equilibrium equation is given by

$$\sigma_{ij,j} + \cancelto{0}{b_i} = 0$$

Written in index notation, the equations for stress become

$$\sigma_{11} = 3x_1^2 + 4x_1x_2 - 8x_2^2$$
$$\sigma_{12} = \frac{1}{2}x_1^2 - 6x_1x_2 - 2x_2^2$$
$$\sigma_{22} = 2x_1^2 + x_1x_2 + 3x_2^2$$
$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

The equilibrium equation can be rewritten as

$$\sigma_{ij,j} = \sum_{j=1}^{3} \sigma_{ij,j} = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3}$$

For $i = 1$,

$$\sigma_{1j,j} = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3}$$

Evaluating the derivatives yields

$$\sigma_{11,1} = 6x_1 + 4x_2$$
$$\sigma_{12,2} = -6x_1 - 4x_2$$
$$\sigma_{13,3} = 0$$

Therefore,

$$\sigma_{1j,j} = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0$$

For $i = 2$,

$$\sigma_{2j,j} = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3}$$

Evaluating the derivatives:

$$\sigma_{21,1} = x_1 - 6x_2$$
$$\sigma_{22,2} = x_1 + 6x_2$$
$$\sigma_{23,3} = 0$$

Therefore,

$$\sigma_{2j,j} = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 2x_1 \neq 0$$

Therefore, equilibrium does not exist for the given stress distribution.

Problem 3.26

Problem Statement

A rope is hung from the ceiling. Let the density of the rope be 2 g/cm^3. Find the stress in the rope.

Solution

The equilibrium equation is given by

$$\sigma_{ij,j} + X_i = 0$$

where X_i is a body force. Let the positive x_1-direction be down. Then, the equation for $i = 1$ is

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + X_1 = 0$$

Assuming that $\sigma_{12} = \sigma_{13} = 0$, the equation becomes

$$\sigma_{11,1} + X_1 = 0$$

The body force is due to gravity. Therefore,

$$X_1 = \rho g$$

Inserting this relation for the body force yields

$$\frac{\partial \sigma_{11}}{\partial x_1} + \rho g = 0$$

Letting $\sigma_{11} = \sigma_x$ and manipulating the equation yields

$$\begin{aligned}\sigma_x &= -\int \rho g \, dx \\ &= -\rho g x + C\end{aligned}$$

At the end of the rope (i.e., $x = L$), $\sigma_x = 0$.

$$\sigma_x = 0 = -\rho g L + C$$

So,

$$C = \rho g L$$

Therefore, the stress in the rope is

$$\boxed{\sigma_x = -\rho g(x - L)}$$

Plugging in the numbers yields

$$\sigma_x = -19620(x - L)$$

Problem 3.27

Problem Statement

Consider a vertical column of an isothermal atmosphere that obeys the gas laws $p/\rho = RT$. or $p = \rho RT$, where ρ is the density of the gas, p is the pressure, R is the gas constant and T is the absolute pressure. The gas is subject to a gravitational acceleration g so that the body force is ρg per unit volume, pointing to the ground. If the pressure at the ground level $z = 0$ is p_o, determine the relation between the pressure and the height z above ground.

Solution

The equilibrium equation is given by

$$\sigma_{ij,j} + X_i = 0$$

Because the properties to not vary in the x or y directions, $\sigma_{12,2} = \sigma_{13,3} = 0$. The equilibrium equation in the z direction, where z is measured positive from the ground up, can be written as

$$\sigma_{11,1} + X_1 = 0$$

$$\frac{\partial \sigma_{11}}{\partial z} + X_1 = 0$$

Since stress is positive outward, and pressure is directed inward,

$$p = -\sigma_{11}$$

Also, the body force is the specific weight of the column of fluid.

$$X_1 = -\rho g$$

The equation becomes

$$-\frac{\partial p}{\partial z} - \rho g = 0$$

Rearranging and inserting the relation for the ideal gas law yields

$$\frac{\partial p}{\partial z} = -\frac{p}{RT} g$$

Manipulating further,

$$\int_{p_0}^{p} \frac{1}{p}\, dp = -\int_0^z \frac{g}{RT}\, dz$$

Evaluating the integrals,

$$\ln(p) - \ln(p_0) = -\frac{gz}{RT}$$

$$\ln\left(\frac{p}{p_0}\right) = -\frac{gz}{RT}$$

Raising both sides to the e,

$$\frac{p}{p_0} = \exp\left(-\frac{gz}{RT}\right)$$

Therefore,

$$\boxed{p = p_0 \exp\left(-\frac{gz}{RT}\right)}$$

Problem 3.29

Problem Statement

Consider a two-dimensional state of stress in a thin plate in which $\tau_{zz} = \tau_{zx} = \tau_{zy} = 0$. The equation of equilibrium acting in the plate in the absence of body force are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Show that these equation are satisfied identically if σ_x, σ_y, and τ_{xy} are derived from an arbitrary function $\Phi(x, y)$ such that

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Solution

This problem is solved by inserting the derivatives of Φ with respect to the coordinates into the equilibrium equations. For the first equilibrium equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \Phi}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial^2 \Phi}{\partial x \partial y}\right) \stackrel{?}{=} 0$$

$$\boxed{\frac{\partial^3 \Phi}{\partial x \partial y^2} - \frac{\partial \Phi^3}{\partial x \partial y^2} \stackrel{\checkmark}{=} 0}$$

For the second equilibrium equation,

$$\frac{\partial}{\partial x}\left(-\frac{\partial^2 \Phi}{\partial x \partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \Phi}{\partial x^2}\right) \stackrel{?}{=} 0$$

$$\boxed{-\frac{\partial^3 \Phi}{\partial x^2 \partial y} + \frac{\partial^3 \Phi}{\partial x^2 \partial y} \stackrel{\checkmark}{=} 0}$$

Therefore, the equations are satisfied.

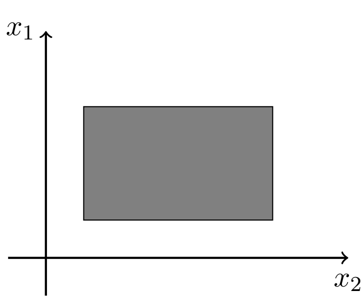
Problem 2

Problem Statement

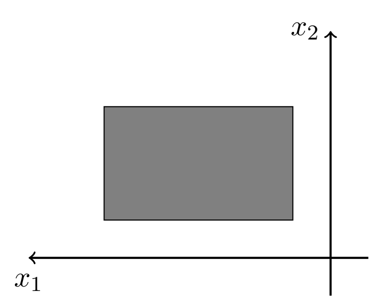
Label the stresses given as

$$\sigma_{ij} = \begin{bmatrix} -4 & -5 \\ -5 & 1 \end{bmatrix}$$

for the following two cases:



(a) Case 1 diagram.



(b) Case 2 diagram.

Solution

$$t_i = \sigma_{ij} n_j$$

For the first case, the components of the traction vector for the upper surface are

$$t = \begin{bmatrix} -4 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$$

For the right surface,

$$t = \begin{bmatrix} -4 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

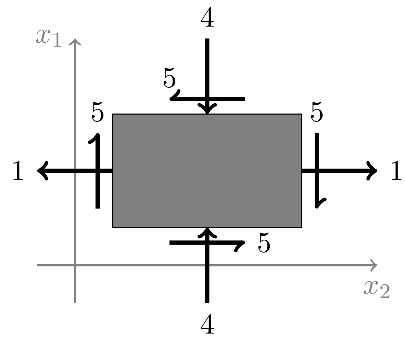


Figure 2: Case 1 stresses.

Following the same procedure for part (b) yields

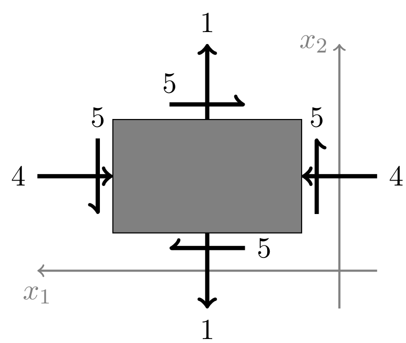


Figure 3: Case 2 stresses.

Looking at the figures closely, it is evident that part (b) is the same as part (a), with a 90 degree rotation.

Problem 2

Problem Statement

For the stress components given as

$$\sigma_{ij} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & -7 \\ 2 & -7 & -2 \end{bmatrix}$$

compute t_i, N_i, S_i, N and S at $(1, 1, 1)$ on the curved surface given by $xyz^3 = 1$.

Solution

The traction force is computed using the following equation

$$t_i = \sigma_{ij} n_j$$

Taking the total derivative of the equation yields

$$yz^3\, dx + xz^3\, dy + 3xyz^2\, dz = 0$$

The above equation can be written as a dot product of two vectors.

$$\begin{Bmatrix} yz^3 \\ xz^3 \\ 3xyz^2 \end{Bmatrix} \begin{Bmatrix} dx & dy & dz \end{Bmatrix} = 0$$

Evaluating the vector on the left at $(1, 1, 1)$,

$$\mathbf{n} = \begin{Bmatrix} 1 \\ 1 \\ 3 \end{Bmatrix}$$

However, the normal vector must be normalized so that it is a unit vector.

$$\|\mathbf{n}\| = \sqrt{1 + 1 + 9} = \sqrt{11}$$

The normalized vector is then

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{11}} \begin{Bmatrix} 1 \\ 1 \\ 3 \end{Bmatrix}$$

The traction vector is then given by

$$\mathbf{t} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & -7 \\ 2 & -7 & -2 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{Bmatrix}$$

$$\boxed{\mathbf{t} = \left\{ \frac{4}{\sqrt{11}}, -\frac{23}{\sqrt{11}}, -\sqrt{11} \right\}}$$

The angle between the traction vector and the normal vector can be found as follows

$$\mathbf{t} \cdot \hat{\mathbf{n}} = |\mathbf{t}||\hat{\mathbf{n}}| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{t} \cdot \hat{\mathbf{n}}}{|\mathbf{t}||\hat{\mathbf{n}}|}\right)$$

$$\boxed{N = \mathbf{t} \cdot \hat{\mathbf{n}} = -\frac{51}{11}}$$

$$|\mathbf{t}| = \sqrt{\frac{666}{11}} \quad \text{and} \quad |\hat{\mathbf{n}}| = 1$$

$$\theta = 127.411^\circ$$

$$N_i = N\hat{\mathbf{n}}$$

$$\boxed{\mathbf{N} = \left\{-\frac{52}{11\sqrt{11}}, -\frac{52}{11\sqrt{11}}, -\frac{156}{11\sqrt{11}}\right\}}$$

$$\mathbf{S} = \mathbf{t} - \mathbf{N}$$

$$\boxed{\mathbf{S} = \left\{\frac{96}{11\sqrt{11}}, -\frac{201}{11\sqrt{11}}, \frac{156}{11\sqrt{11}} - \sqrt{11}\right\}}$$

$$\boxed{S = \sqrt{\left(\frac{156}{11\sqrt{11}} - \sqrt{11}\right)^2 + \frac{49617}{1331}}}$$

Part II

ME 5312 - Homework 3

James Grisham

```
In[92]:= ClearAll["Global`*"]
```

Problem 1

```
In[93]:= rvec = {x[1], x[2], x[3]};

In[94]:= r = Sqrt[Sum[x[i]^2, {i, 1, 3}]];

In[95]:= div[f_] := Sum[D[f[[i]], x[i]], {i, 1, 3}]

In[96]:= div[rvec]

Out[96]= 3

In[97]:= div[r^n * rvec] // Simplify
```

Out[97]= $(3+n)\left(x[1]^2+x[2]^2+x[3]^2\right)^{n/2}$

```
In[99]:= curl[f_] :=
          Table[Sum[Signature[{i, j, k}] * D[f[[j]], x[k]], {j, 1, 3}, {k, 1, 3}], {i, 1, 3}]

In[100]:= curl[r^n * rvec]

Out[100]= {0, 0, 0}

In[101]:= laplacian[f_] := Sum[D[f, {x[i], 2}], {i, 1, 3}]

In[102]:= laplacian[r^n] // FullSimplify
```

Out[102]= $n\,(1+n)\left(x[1]^2+x[2]^2+x[3]^2\right)^{-1+\frac{n}{2}}$

Problem 2

```
In[103]:= uu = {u, v, w};

In[104]:= xx = {x, y, z};

In[109]:= Table[Sum[Dt[uu[[i]], {xx[[k]], 2}], {k, 1, 3}] , {i, 1, 3}]

Out[109]= {Dt[u, {x, 2}] + Dt[u, {y, 2}] + Dt[u, {z, 2}],
          Dt[v, {x, 2}] + Dt[v, {y, 2}] + Dt[v, {z, 2}], Dt[w, {x, 2}] + Dt[w, {y, 2}] + Dt[w, {z, 2}]}
```

In[112]:=
```
Table[Sum[Dt[Dt[uu[[k]], xx[[k]]], xx[[i]]], {k, 1, 3}], {i, 1, 3}]
```

Out[112]=
```
{Dt[u, {x, 2}] + Dt[v, x, y] + Dt[w, x, z],
 Dt[v, {y, 2}] + Dt[u, x, y] + Dt[w, y, z], Dt[w, {z, 2}] + Dt[u, x, z] + Dt[v, y, z]}
```

In[117]:=
```
Table[G * (Sum[Dt[uu[[i]], {xx[[k]], 2}], {k, 1, 3}] +
      1 / (1 - 2 * ν) * Sum[Dt[Dt[uu[[k]], xx[[k]]], xx[[i]]], {k, 1, 3}]) +
   X[i] == ρ * Dt[uu[[i]], {t, 2}], {i, 1, 3}]
```

Out[117]=

$$\left\{G\left(Dt[u,\{x,2\}]+Dt[u,\{y,2\}]+Dt[u,\{z,2\}]+\frac{Dt[u,\{x,2\}]+Dt[v,x,y]+Dt[w,x,z]}{1-2\nu}\right)+X[1]==\rho\,Dt[u,\{t,2\}],\ G\left(Dt[v,\{x,2\}]+Dt[v,\{y,2\}]+Dt[v,\{z,2\}]+\frac{Dt[v,\{y,2\}]+Dt[u,x,y]+Dt[w,y,z]}{1-2\nu}\right)+X[2]==\rho\,Dt[v,\{t,2\}],\ G\left(Dt[w,\{x,2\}]+Dt[w,\{y,2\}]+Dt[w,\{z,2\}]+\frac{Dt[w,\{z,2\}]+Dt[u,x,z]+Dt[v,y,z]}{1-2\nu}\right)+X[3]==\rho\,Dt[w,\{t,2\}]\right\}$$