AE 5331: Analytic Methods Engineering
Homework 1

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Problem 1

Problem Statement

Determine the following limits:

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

(b) $\lim_{x \to 0} \frac{\sin(2x)}{x}$

(c) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

(d) $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

Solution

Part A

Applying the limit directly yields

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0}$$

which is indeterminate. Therefore, this problem is a candidate for l'Hopital's rule.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\left(\sin(x)\right)}{\frac{d}{dx}\left(x\right)} = \lim_{x \to 0} \cos(x)$$

Therefore,

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Part B

Applying the limit directly yields

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=\frac{0}{0}$$

Therefore, this problem is a candidate for l'Hopital's rule.

$$\lim_{x\to 0}\frac{\sin(2x)}{x}=\lim_{x\to 0}2\cos(2x)$$

$$\boxed{\lim_{x\to 0}\frac{\sin(2x)}{x}=2}$$

Part C

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to\infty}e^{x\ln\left(1+\frac{1}{x}\right)}$$

Because e is constant, we will look at the limit of the power first.

$$\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}=\frac{\ln(1)}{0}=\frac{0}{0}$$

which is indeterminate. So, this problem is a candidate for l'Hopital's rule.

$$\lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cancel{(-x^{-2})}}{\cancel{-x^{-2}}}=\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}=1$$

Therefore,

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e}$$

Part D

Using the same approach as before,

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=e^{x\ln(1-\frac{1}{x})}$$

Again, since e is a constant, we can apply the limit to the exponent.

$$\lim_{x\to\infty}x\ln\left(1-\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}=\frac{0}{0}$$

Therefore, this problem is a candidate for l'Hoptial's rule.

$$\lim_{x\to\infty}x\ln\left(1-\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\frac{1}{1-\frac{1}{x}}\cancel{(x^{-2})}}{-\cancel{x^{-2}}}=\lim_{x\to\infty}\frac{-1}{1-\frac{1}{x}}=-1$$

So, the limit is

$$\boxed{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=\frac{1}{e}}$$

Problem 2

Problem Statement

Examine continuity and differentiability of the following functions at $x = 0$:

(a) $f(x) = \begin{cases} \sin(x), & x < 0 \\ x, & x \geqslant 0 \end{cases}$

(b) $f(x) = |\sin(x)|$

(c) $f(x) = \begin{cases} x^2, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$

Solution

Part A

According to the definition of continuity, a function $f(x)$ is continuous at $x = x_0$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

For differentiability, we must show that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. The value of $f(x_0)$ is

$$f(x_0) = f(0) = 0$$

Applying the limit from the left,

$$\lim_{x \to 0^-} \sin(x) = 0$$

From the right,

$$\lim_{x \to 0^+} x = 0$$

Since the limits approach $f(0)$, the function is continuous. Now, checking for differentiability from the left,

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{\sin(h)}{h} = \frac{0}{0}$$

Applying l'Hopital's rule,

$$\lim_{h \to 0^-} \frac{\sin(h)}{h} = \lim_{h \to 0^-} \cos(h) = 1$$

From the right,

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Therefore, the function is both, continuous and differentiable about $x = 0$.

Part B

The function is

$$f(x) = |\sin(x)|$$

Testing for continuity,

$$\lim_{x \to 0} |\sin(x)| = 0$$

This applies to the limit from either side. Both are equal to zero. Testing for differentiability from the left,

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-\sin(h) + \sin(0)}{h} = \lim_{h \to 0} \frac{-\sin(h)}{h} = -1$$

From the right,

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sin(h) - \sin(0)}{h} = \lim_{h \to 0^+} \frac{\sin(h)}{h} = 1$$

Therefore, the function is continuous, but not differentiable.

Part C

$$f(x) = \begin{cases} x^2, & x \geqslant 0 \\ -x, & x < 0 \end{cases}$$

Testing for continuity from the left,

$$\lim_{x \to 0^-} (-x) = 0$$

From the right,

$$\lim_{x \to 0^+} x^2 = 0$$

Therefore, the function is continuous. Testing for differentiability from the right:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2}{h} = 0$$

Testing for differentiability from the left,

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = -1$$

Therefore, the function is continuous, but not differentiable.

Problem 3

Problem Statement

Determine whether the series in "More examples" and "Exercise of infinite series" of Lecture 3 (whose notes are available on the course website) are convergent or divergent, and prove it. Note: if a problem appears twice, you only need to solve it once.

The series are:

(a) $\sum_{n=1}^{\infty}(n+3)^{-3/2}$

(b) $\sum_{n=1}^{\infty}\frac{n}{n^2+3\ln(n)}$

(c) $\sum_{n=1}^{\infty}n^{-100}$

(d) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$

(e) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+10}\right)$

(f) $\sum_{n=1}^{\infty}\left(\frac{\cos(n)}{2n-1}\right)^2$

(g) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

(h) $\sum_{n=1}^{\infty}\frac{n+(\cos(n))^2}{n^2+4}$

(i) $\sum_{n=1}^{\infty}e^{-nx}$

(j) $\sum_{n=1}^{\infty}\ln\left(2+\frac{2}{n}\right)$

(k) $\sum_{n=1}^{\infty}\left(\frac{n^2+2n-1}{n^4+3}\right)^{\frac{3}{2}}$

(l) $\sum_{n=1}^{\infty}(-1)^n\ln\left(1+\frac{1}{\sqrt{n}}\right)$

(m) $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2}$

(n) $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}$

(o) $\sum_{n=1}^{\infty}\ln\left(\frac{n^2-1}{n^2+1}\right)$

Solution

Part A

The series is

$$\sum_{n=1}^{\infty}(n+3)^{-3/2}$$

Convergence of this series can be determined using the comparison test.

> **Comparison Test** Let $\sum a_n$ be a series with no negative terms.
>
> (a) $\sum a_n$ converges if there is a convergent series $\sum b_n$ with $a_n \leqslant b_n$ for all $n > N$, for some integer N.
>
> (b) $\sum a_n$ diverges if there is a divergent series of nonnegative terms $\sum c_n$ with $a_n \geqslant c_n$ for all $n > N$, for some integer N.

Letting

$$a_n = \frac{1}{(n+3)^{3/2}} \quad \text{and} \quad b_n = \frac{1}{n^{3/2}}$$

The first sequence a_n is always less than b_n because of the added constant in the denominator. Therefore,

$$a_n \overset{\checkmark}{\leqslant} b_n$$

$\sum b_n$ corresponds to a p-series with $p > 1$. Therefore, $\sum b_n$ is convergent. Since $\sum b_n$ is convergent, by the comparison test, a_n must also be a convergent series.

Part B

$$\sum_{n=1}^{\infty}\frac{n}{n^2+3\ln(n)}$$

In this case, the limit comparison test will be used.

Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geqslant N$ (where N is an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Let

$$a_n = \frac{n}{n^2 + 3\ln(n)}$$

and

$$b_n = \frac{1}{n}$$

Evaluating the limit yields

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+3\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 + 3\ln(n)} = \lim_{n\to\infty} \frac{1}{1 + \frac{3\ln(n)}{n^2}}$$

Looking at the limit of the second term in the denominator more closely,

$$\lim_{n\to\infty} \frac{3\ln(n)}{n^2} = \frac{\infty}{\infty}$$

which is indeterminate. Therefore, this limit is a candidate for l'Hopital's rule.

$$\lim_{n\to\infty} \frac{3\ln(n)}{n^2} = \lim_{n\to\infty} \frac{3}{2n^2} = 0$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \tag{1}$$

Because the value of the limit in (1) is greater than zero and finite, $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Because we already know that b_n diverges (it is a power series with $p = 1$), $\sum a_n$ must also diverge.

Part C

$$\sum_{n=1}^{\infty} n^{-100}$$

This series is a p-series with $p = 100$. Therefore, the series is convergent.

Part D

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$$

Letting

$$a_n = 1+\frac{1}{n^2}$$

and

$$b_n = \frac{1}{n^0}$$

where $\sum b_n$ is a divergent p-series. The first series ($\sum a_n$) will always be greater than the second series ($\sum b_n$). Therefore, by the comparison test, this series diverges.

Part E

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+10}\right)=\sum_{n=1}^{\infty}\frac{10}{n^2+10n}$$

Letting

$$a_n = \frac{10}{n^2+10n}$$

and

$$b_n = \frac{1}{n^2}$$

where b_n is a convergent p-series. Using the limit comparison test,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{10}{n^2+10n}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{10}{1+\frac{10}{n}}=10$$

Since

$$\lim_{n\to\infty}\frac{a_n}{b_n}=c>0$$

two situations are possible: (1) $\sum a_n$ and $\sum b_n$ both converge or (2) $\sum a_n$ and $\sum b_n$ both diverge. Since we know that $\sum b_n$ is a convergent p-series, $\sum a_n$ must also be convergent.

Part F

$$\sum_{n=1}^{\infty}\left(\frac{\cos(n)}{2n-1}\right)^2$$

Letting

$$a_n = \left(\frac{\cos(n)}{2n-1}\right)^2$$

and

$$b_n = \frac{1}{n^2}$$

where $\sum b_n$ is a convergent p-series, and applying the limit comparison test,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{\cos(n)}{2n-1}\right)^2}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2\cos^2(n)}{4n^2-4n+1} = \lim_{n\to\infty} \frac{\cos^2(n)}{4-\frac{4}{n}+\frac{1}{n^2}} \tag{2}$$

The final limit in (2) could be anything from 0 to 1/4 (because of the $\cos^2(n)$ term). Since $\sum b_n$ is a convergent series, and since a_n is either equal to zero or greater than zero, a_n must also be a convergent series.

Part G

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

The convergence of this series can be determined using the alternating series test.

Alternating series Test The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. $u_n \geqslant u_{n+1}$ for all $n \geqslant N$, for some integer N.
3. $u_n \to 0$.

In this case,

$$u_n = \frac{1}{n}$$

So, the first condition is satisfied. Now, the second condition (with $N = 1$):

$$u_n \overset{?}{\geqslant} u_{n+1} \;\forall\; n \geqslant N$$

$$\frac{1}{n} \overset{\checkmark}{\geqslant} \frac{1}{n+1}$$

Therefore, the second condition is also satisfied.

$$u_n \to 0 \text{ as } n \to \infty?$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, all three conditions of the alternating series test are met and the series is convergent.

Part H

$$\sum_{n=1}^{\infty} \frac{n+\cos^2(n)}{n^2+4}$$

Letting

$$a_n = \frac{n+\cos^2(n)}{n^2+4}$$

and

$$b_n = \frac{1}{n}$$

where $\sum b_n$ is a Harmonic series (divergent), and applying the limit comparison test,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+\cos^2(n)}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n\cos^2(n)}{n^2+4} = \lim_{n\to\infty} \frac{1+\frac{\cos^2(n)}{n}}{1+\frac{4}{n^2}} = 1$$

Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

$\sum a_n$ and $\sum b_n$ either both converge or both diverge. Since $\sum b_n$ is a divergent series, $\sum a_n$ must also be a divergent series.

Part I

$$\sum_{n=1}^{\infty} e^{-nx}$$

This is a geometric series. In general, a geometric series can be written as

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1 \tag{3}$$

If $|e^{-x}| < 1$, or equivalently, $|e^{x}| > 1$, the series converges to

$$\sum_{n=1}^{\infty} \left(e^{-x}\right)^n = \frac{1}{1-e^{-x}} - 1 \tag{4}$$

where the -1 was added to account for the different indices in (3) and (4).

Part J

$$\sum_{n=1}^{\infty} \ln\left(2+\frac{2}{n}\right)$$

This series diverges. Divergence can be proved using the n-th term test.

nth-Term Test for Divergence $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

$$\lim_{n\to\infty} \ln\left(2+\frac{2}{n}\right) = \ln(2) \neq 0$$

Therefore, this series diverges.

Part K

$$\sum_{n=1}^{\infty}\left(\frac{n^2+2n-1}{n^4+3}\right)^{\frac{3}{2}}$$

Convergence can be determined using the limit comparison test. Letting

$$a_n = \left(\frac{n^2+2n-1}{n^4+3}\right)^{\frac{3}{2}}$$

and

$$b_n = \frac{1}{n^3}$$

where Σb_n is a convergent p-series, and applying the limit

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\frac{n^2+2n-1}{n^4+3}\right)^{\frac{3}{2}}}{\frac{1}{n^3}} = \lim_{n\to\infty}\frac{\left(\frac{n^2+2n-1}{n^4+3}\right)^{\frac{3}{2}}}{\left(\left(\frac{1}{n^3}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}}} = \lim_{n\to\infty}\left(\frac{n^4+2n^3-n^2}{n^4+3}\right)^{\frac{3}{2}}$$

$$\lim_{n\to\infty}\left(\frac{n^4+2n^3-n^2}{n^4+3}\right)^{\frac{3}{2}} = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}-\frac{1}{n^2}}{1+\frac{3}{n^4}}\right)^{\frac{3}{2}} = 1$$

Therefore, Σa_n and Σb_n either both converge or both diverge. Since we already know that Σb_n is a convergent p-series, we can say that Σa_n must also be a convergent series.

Part L

$$\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{\sqrt{n}}\right)$$

The convergence of this series can be determined using the alternating series test. In this case,

$$u_n = \ln\left(1+\frac{1}{\sqrt{n}}\right)$$

The first condition is that all of the u_ns are positive. This condition is satisfied because the argument of the natural log function is always greater than or equal to one.

The second condition is that

$$u_n \geqslant u_{n+1} \;\forall\; n \geqslant N$$

where N is just some integer. As n increases, the argument of the natural log decreases. Therefore, u_n decrease. Thus, the second condition is satisfied.

The third condition requires that

$$\lim_{n\to\infty} u_n \to 0$$

Applying this,

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{\sqrt{n}}\right) = 0$$

Because all three conditions are met, the series is convergent.

Part M

$$\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$$

The convergence of this series can be determined using the limit comparison test. Letting

$$a_n = \frac{1}{x^2+n^2}$$

and

$$b_n = \frac{1}{n^2}$$

where Σb_n is a convergent p-series, and applying the limit,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{x^2+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\frac{x^2}{n^2}+1} = 1$$

Therefore, as long as x is finite, either both Σa_n and Σb_n converge or diverge. Since we know that Σb_n is a convergent p-series, Σa_n must also be a convergent series.

Part N

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

The convergence of this series can be determined using the alternating series test with

$$u_n = \frac{1}{2^n}$$

The first condition requires that all values of u_n are positive. This condition is satisfied.

The second condition requires

$$u_n \geqslant u_{n+1} \ \forall n \geqslant N$$

where N is some integer. This condition is also satisfied.

$$\frac{1}{2^n} \overset{?}{\geqslant} \frac{1}{2^{n+1}} \Rightarrow 2 \overset{\checkmark}{\geqslant} 1$$

The third condition requires that $u_n \to 0$.

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Therefore, the series is convergent.

Part O

$$\sum_{n=1}^{\infty} \ln\left(\frac{n^2-1}{n^2+1}\right) = \sum_{n=1}^{\infty}\left[\ln\left(1-\frac{1}{n^2}\right) - \ln\left(1+\frac{1}{n^2}\right)\right] = \sum_{n=1}^{\infty} \ln\left(1-\frac{1}{n^2}\right) - \sum_{n-1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$$

Simplifying further,

$$\sum_{n=1}^{\infty} \ln\left(1-\frac{1}{n^2}\right) - \sum_{n-1}^{\infty} \ln\left(1+\frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \ln\left(1-\frac{1}{n^2}\right) + \sum_{n-1}^{\infty} \ln\left(\frac{n^2}{n^2+1}\right)$$

Both of the series are convergent. The sum of two convergent series is a convergent series. Therefore, the series is convergent.

Exercises 13.4

Problem 1

Let $f(x, y) = \sin(x^4 + 3y)$, where $x = 5t$ and $y = t^2 + 1$, and denote $f(x(t), y(t)) = F(t)$. Evaluate dF/dt using the chain rule,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

NOTE: Actually, the above equation is not the end of the "chain differentiation story," for in computing $\partial f/\partial x$, we set $x^4 + 3y = u$, so that, again applying chain differentiation,

$$\frac{\partial f}{\partial x} = \frac{d}{du}(\sin u)\frac{\partial u}{\partial x} = \text{etc.}$$

and similarly for $\partial f/\partial y$.

$$\frac{\partial f}{\partial x} = \cos(x^4 + 3y)(4x^3)$$

$$\frac{\partial f}{\partial y} = \cos(x^4 + 3y)(3)$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = 2t$$

Therefore,

$$\frac{dF}{dt} = 20(5t)^3 \cos((5t)^4 + 3(t^2 + 1)) + 6t \cos((5t)^4 + 3(t^2 + 1))$$

Simplifying further,

$$\boxed{\frac{dF}{dt} = (2500t^3 + 6t)\cos(625t^4 + 3t^2 + 3)}$$

Problem 2(b)

Let $f(x,y) = e^{xy}$, and denote $f(x(t), y(t)) = F(t)$. Evaluate dF/dt in each case, using the chain rule.

$$x(t) = \sqrt{t+1}$$
$$y(t) = \cos(t)$$

The chain rule is

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = ye^{xy}$$
$$\frac{\partial f}{\partial y} = xe^{xy}$$
$$\frac{dx}{dt} = \frac{1}{2}(t+1)^{-1/2}$$
$$\frac{dy}{dt} = -\sin(t)$$

Therefore,

$$\frac{dF}{dt} = \cos(t)e^{\sqrt{t+1}\cos(t)}\frac{1}{2\sqrt{t+1}} - \sqrt{t+1}e^{\sqrt{t+1}\cos(t)}\sin(t)$$

$$\boxed{\frac{dF}{dt} = \left[\frac{\cos(t)}{2\sqrt{t+1}} - \sqrt{t+1}\sin(t)\right]e^{\sqrt{t+1}\cos(t)}}$$

Problem 2(d)

$$x(t) = \ln(t)$$
$$y(t) = t$$

Therefore,

$$\frac{dx}{dt} = \frac{1}{t}$$
$$\frac{dy}{dt} = 1$$

Now,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Thus,

$$\boxed{\frac{dF}{dt} = e^{t\ln(t)}\left(1 + \ln(t)\right)}$$

However,

$$e^{t\ln(t)} = e^{\ln(t^t)} = t^t$$

Using this,

$$\boxed{\frac{dF}{dt} = t^t\left(1 + \ln(t)\right)}$$

Problem 2(f)

$$x(t) = 3t - 1$$
$$y(t) = 2t + 5$$

$$\frac{dx}{dt} = 3$$
$$\frac{dy}{dt} = 2$$
$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Inserting the derivatives into the chain rule and simplifying yields

$$\boxed{\frac{dF}{dt} = (12t + 13)e^{6t^2+13t-5}}$$

Exercises 13.5

Problem 1

Expand the given function about the indicated point a, through third order terms. NOTE: $(x-a)^n$ is of nth order.

The general equation for the Taylor series of a function of one variable is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Expanding through third order terms yields

$$f(x)|_{x=a} \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

0.0.1 Part a

$$f(x) = e^{-2x} \text{ about } a = 0$$

$$f'(x) = -2e^{-2x}$$
$$f''(x) = 4e^{-2x}$$
$$f'''(x) = -8e^{-2x}$$

Thus, the expansion about $x = 0$ is

$$\boxed{e^{-2x}|_0 \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3}$$

Part b

$$f(x) = e^{-2x} \text{ about } a = 5$$

$$\begin{aligned} f'(x) &= -2e^{-2x} \\ f''(x) &= 4e^{-2x} \\ f'''(x) &= -8e^{-2x} \end{aligned}$$

Thus, the expansion about $x = 5$ is

$$\boxed{e^{-2x}|_5 \approx e^{-10}\left(1 - 2(x-5) + 2(x-5)^2 - \frac{4}{3}(x-5)^3\right)}$$

Part c

$$f(x) = e^{-2x} \text{ about } a = -3$$

$$\begin{aligned} f'(x) &= -2e^{-2x} \\ f''(x) &= 4e^{-2x} \\ f'''(x) &= -8e^{-2x} \end{aligned}$$

Thus, the expansion about $x = -3$ is

$$\boxed{e^{-2x}|_{-3} \approx e^{6}\left(1 - 2(x+3) + 2(x+3)^2 - \frac{4}{3}(x+3)^3\right)}$$

Part d

$$f(x) = \ln(x) \text{ about } a = 2$$

$$\begin{aligned} f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \end{aligned}$$

Thus, the expansion about $x = 2$ is

$$\boxed{\ln(x)|_2 \approx \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3}$$

Part e

$$f(x) = \frac{1}{1+x^2} \text{ about } a = 1$$

$$\begin{aligned} f'(x) &= -2x(1+x^2)^{-2} \\ f''(x) &= 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2} \\ f'''(x) &= -48x^3(1+x^2)^{-4} + 24x(1+x^2)^{-3} \end{aligned}$$

Thus, the expansion about $x = 1$ is

$$\boxed{\left.\frac{1}{1+x^2}\right|_1 \approx \frac{1}{2} - \frac{x-1}{2} + \frac{(x-1)^2}{4}}$$

The third order term is zero because $f'''(1) = 0$.

Part f

$$f(x) = \frac{1}{1+x^2} \text{ about } a = -1$$

$$\begin{aligned} f'(x) &= -2x(1+x^2)^{-2} \\ f''(x) &= 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2} \\ f'''(x) &= -48x^3(1+x^2)^{-4} + 24x(1+x^2)^{-3} \end{aligned}$$

Thus, the expansion about $x = -1$ is

$$\boxed{\left.\frac{1}{1+x^2}\right|_{-1} \approx \frac{1}{2} + \frac{x+1}{2} + \frac{(x+1)^2}{4}}$$

The third order term is zero because $f'''(1) = 0$.

Part g

$$f(x) = \sin(x) \text{ about } a = 2$$

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \end{aligned}$$

Thus, the expansion about $x = 2$ is

$$\boxed{\sin(x)|_2 \approx \sin(2) + (x-2)\cos(2) - \frac{(x-2)^2}{2}\sin(2) - \frac{(x-2)^3}{6}\cos(2)}$$

Part h

$$f(x) = \cos(2x) \text{ about } a = \pi$$

$$\begin{aligned} f'(x) &= -2\sin(2x) \\ f''(x) &= -4\cos(2x) \\ f'''(x) &= 8\sin(2x) \end{aligned}$$

Thus, the expansion about $x = \pi$ is

$$\boxed{\cos(2x)|_{\pi} \approx 1 - 2(x-\pi)^2}$$

Part i

$$f(x) = x(x-1)^2 = x^3 - 2x^2 + x \text{ about } a = 1$$

$$\begin{aligned} f'(x) &= 3x^2 - 4x + 1 \\ f''(x) &= 6x - 4 \\ f'''(x) &= 6 \end{aligned}$$

Thus, the expansion about $x = 1$ is

$$\boxed{x(x-1)^2|_1 \approx (x-1)^2 + (x-1)^3}$$

Simplifying this equation yields the original equation.

Part j

$$f(x) = x^3(x^4 - 1) + 5 = x^7 - x^3 + 5 \text{ about } a = 0$$

$$\begin{aligned} f'(x) &= 7x^6 - 3x^2 \\ f''(x) &= 42x^5 - 6x \\ f'''(x) &= 210x^4 - 6 \end{aligned}$$

Thus, the expansion about $x = 0$ is

$$\boxed{x^3(x^4-1) + 5|_0 \approx 5 - x^3}$$

Problem 2(b)

Obtain the first four nonvanishing terms in the Taylor series of the given function about $x = 0$.

$$f(x) = \frac{1}{2 + x^{10}}$$

The general equation for a Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Solving this problem requires a lot of differentiation. The problem was solved with the help of a CAS, specifically, SymPy. The first derivative is

$$f'(x) = -\frac{10x^9}{\left(x^{10}+2\right)^2}$$

However, $f'(0) = 0$ which means that the first order term vanishes. In fact, all the terms vanish until the 10th order term which is

$$f^{(10)}(0) = -907200$$

The symbolic expression was too large to fit on the page. The next derivative that is nonzero is the $20th$ order derivative which is

$$f^{(20)}(0) = 304112751022080000$$

and so on up until $30th$ order terms. Thus, the Taylor series expansion about $x = 0$ is given by

$$\boxed{\left.\frac{1}{2+x^{10}}\right|_0 = \frac{1}{2} - \frac{x^{10}}{4} + \frac{x^{20}}{8} - \frac{x^{30}}{16} + \mathcal{O}\left(x^{40}\right)}$$

Problem 2(d)

The function given in this problem is

$$f(x) = \cos(x^{20})$$

$$\begin{aligned}
f'(x) &= -20x^{19}\sin\left(x^{20}\right)\\
f''(x) &= -20x^{18}\left(20x^{20}\cos\left(x^{20}\right) + 19\sin\left(x^{20}\right)\right)\\
f'''(x) &= 40x^{17}\left(200x^{40}\sin\left(x^{20}\right) - 570x^{20}\cos\left(x^{20}\right) - 171\sin\left(x^{20}\right)\right)\\
\vdots &= \qquad \vdots
\end{aligned}$$

All of the derivatives evaluated at $x = 0$ are zero up until the 40^{th} order term. Continuing the differentiation until four nonvanishing terms are found yields the following:

$$\boxed{\cos(x^{20})|_0 = 1 - \frac{x^{40}}{2} + \frac{x^{80}}{24} - \frac{x^{120}}{720} + \mathcal{O}\left(x^{138}\right)}$$

Exercises 13.6

Problem 1(b)

Given $f(x,y) = 0$ and a point (x_0, y_0) such that $f(x_0, y_0) = 0$, see if the conditions of Theorem 13.6.1 are met. If so, develop the implicit function $y(x)$ in a Taylor series about x_0, through second order terms, as we did in Example 4.

> **Theorem 13.6.1** *Implicit Function Theorem*
> Let $f(x,y) = 0$ be satisfied by a pair of real numbers x_0, y_0 so that $f(x_0, y_0) = 0$, and suppose that $f(x,y)$ is C^1 in some neighborhood of (x_0, y_0) with
>
> $$\frac{\partial f(x_0, y_0)}{\partial y} \neq 0$$
>
> Then $f(x,y) = 0$ uniquely implies a function $y(x)$ in some neighborhood N of x_0 such that $y(x_0) = y_0$, where $y(x)$ is differentiable in N. The function $f(x,y)$ being C^1 means that the first-order partial derivatives f_x and f_y are continuous. Also, the neighborhood N of x_0 is an open *interval* on the x-axis, whereas the neighborhood of (x_0, y_0) is an open disk in the x, y plane.

For this problem,

$$f(x,y) = x^2 + 4y^2 - 4 = 0; \quad (x_0, y_0) = (0, 1)$$

The implicit function $y(x)$ exists if $f_y(x_0, y_0) \neq 0$. So,

$$f_y = 8y$$

and

$$f_y(0,1) = 8 \neq 0$$

Therefore, the implicit function exists.

To find the implict function using a Taylor series through second order terms, the first and second order derivatives of the implicit function must be determined. The first and second derivatives are given by

$$y'(x) = -\frac{f_x(x, y(x))}{f_y(x, y(x))}$$
$$y''(x) = \frac{2 f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}}{f_y^3}$$

The derivatives are

$$f_x = 2x$$
$$f_{xx} = 2$$
$$f_y = 8y$$
$$f_{yy} = 8$$
$$f_{xy} = 0$$

Thus,

$$y'(x) = -\frac{x}{4y}$$

and

$$y''(x) = -\frac{x^2 + 4y^2}{16y^3}$$

The Taylor series expansion of the implicit function about (x_0, y_0) is

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2}(x - x_0)^2 + \ \dots$$

The value $y(x_0)$ is determined by evaluating $f(x, y)$ at x_0.

$$y(x_0) = 1$$

Evaluating $y'(x_0)$ yields

$$y'(x_0) = 0$$

Next, $y''(x_0)$,

$$y''(x_0) = -\frac{1}{8}$$

Therefore, the Taylor series expansion is

$$\boxed{y(x) \approx 1 - \frac{1}{16}x^2}$$

Problem 1(e)

$$f(x, y) = x(\cos(\pi y) + 1) + (x^3 + 8)y = 0; \quad (-2, 1)$$

The implicit function exists if $f_y(x_0, y_0) \neq 0$.

$$f_y = x^3 - \pi x \sin(\pi y) + 8$$

and

$$f_y(-2, 1) = 0$$

Therefore, the implicit function does not exist.

Problem 1(g)

$$f(x, y) = x - y + \sin(y) = 0$$

The partial of f wrt y is

$$f_y = \cos(y) - 1$$

$$f_y(0, 0) = \cos(0) - 1 = 0$$

Therefore, the implicit function does not exist.

Problem 2(a)

In each case, find $y'(x)$ and $y''(x)$.

$$f(x,y) = xy - y^3 = 1$$

which can be rewritten as

$$f(x,y) = xy - y^3 - 1 = 0$$

The derivatives are

$$\begin{aligned} f_x &= y \\ f_{xx} &= 0 \\ f_y &= x - 3y^2 \\ f_{yy} &= -6y \\ f_{xy} &= 1 \end{aligned}$$

Now, $y'(x)$ and $y''(x)$ are

$$y'(x) = -\frac{f_x}{f_y} \tag{5}$$

$$y''(x) = \frac{2f_x f_y f_{xy} - f_x^2 f_{yy} - f_y^2 f_{xx}}{f_y^3} \tag{6}$$

Inserting the derivatives and simplifying yields

$$\begin{aligned} y'(x) &= -\frac{y}{x - 3y^2} \\ y''(x) &= \frac{2xy}{(x - 3y^2)^3} \end{aligned}$$

Alternatively, this problem can be solved using direct implicit differentiation. For example, the first derivative is

$$xy' + y - 3y^2 y' = 0$$

Solving for y' yields

$$y'(x) = -\frac{y}{x - 3y^2}$$

which is the same as the previous answer. The same holds for the second derivative.

Problem 2(f)

$$f(x,y) = y\cos(y) - x^3 = 0$$

The derivatives are

$$\begin{aligned} f_x &= -3x^2 \\ f_{xx} &= -6x \\ f_y &= -y\sin(y) + \cos(y) \\ f_{yy} &= -y\cos(y) + 2\sin(y) \\ f_{xy} &= 0 \end{aligned}$$

Inserting these derivatives into (5) and (6),

$$y'(x) = -\frac{3x^2}{y\sin(y) - \cos(y)}$$
$$y''(x) = -\frac{3x}{(y\sin(y) - \cos(y))^3}\left(3x^3(y\cos(y) + 2\sin(y)) + 2(y\sin(y) - \cos(y))^2\right)$$

Exercises 13.7

Problem 1(b)

The given function has a critical point at $x = 1$. Classify it as a local maximum, local minimum or horizontal inflection point.

$$f(x) = 3(x-1)^4 + 5$$

This problem can be solved using Theorem 13.7.2 from the text.

Theorem 13.7.2 *Maximum, Minimum Horizontal Inflection Point*
Suppose that

$$f'(x) = f''(x) = \ldots = f^{(n-1)}(x) = 0,$$

but $f^{(n)}(x) \neq 0$, and that $f^{(n)}(x)$ is continuous in some neighborhood of x, where $n \geqslant 2$. If n is even and $f^{(n)}(x) < 0$, then f has a local maximum at x. If n is even and $f^{(n)}(x) > 0$, then f has a local minimum at x. If n is odd, then f has a horizontal inflection point at x.

Differentiating $f(x)$ and evaluating at $x = 1$ yields

$$\begin{aligned}
f'(x) &= 12x^3 - 36x^2 + 36x - 12 \\
f'(1) &= 0 \\
f''(x) &= 36x^2 - 72x + 36 \\
f''(1) &= 36 - 72 + 36 = 0 \\
f'''(x) &= 72x - 72 \\
f'''(1) &= 0 \\
f^{(4)}(x) &= 72 \\
f^{(4)}(1) &= 72
\end{aligned}$$

Because n is even, and $f^{(n)}(x) > 0$, $f(x)$ has a local minimum at $x = 1$.

Problem 1(d)

$$f(x) = (x+1)(x-3)(1-x)^3$$

Taking derivatives and evaluating them at $x = 1$ yields

$$\begin{aligned} f'(x) &= -5x^4 + 20x^3 - 18x^2 - 4x + 7 \\ f'(1) &= 0 \\ f''(x) &= -20x^3 + 60x^2 - 36x - 4 \\ f''(1) &= 0 \\ f'''(x) &= -60x^2 + 120x - 36 \\ f'''(1) &= 24 \end{aligned}$$

Because n is odd, $f(x)$ has a horizontal inflection point at $x = 1$.

Problem 1(f)

$$f(x) = \exp[8(x-1)^5]$$

Taking derivatives and evaluating them at $x = 1$ yields

$$\begin{aligned} f'(x) &= 40\,(x-1)^4\, e^{8(x-1)^5} \\ f'(1) &= 0 \\ f''(x) &= 160\,(x-1)^3 \left(10\,(x-1)^5 + 1\right) e^{8(x-1)^5} \\ f''(1) &= 0 \\ f'''(x) &= 160\,(x-1)^2 \left(400\,(x-1)^{10} + 120\,(x-1)^5 + 3\right) e^{8(x-1)^5} \\ f'''(1) &= 0 \\ f^{(4)}(x) &= 320\,(x-1) \left(8000\,(x-1)^{15} + 4800\,(x-1)^{10} + 480\,(x-1)^5 + 3\right) e^{8(x-1)^5} \\ f^{(4)}(1) &= 0 \\ f^{(5)}(x) &= 320 \left(320000\,(x-1)^{20} + 320000\,(x-1)^{15} + 72000\,(x-1)^{10} + 3000\,(x-1)^5 + 3\right) e^{8(x-1)^5} \\ f^{(5)}(1) &= 960 \end{aligned}$$

Since n is odd, $f(x)$ has a horizontal inflection point at $x = 1$.

Problem 1(g)

$$f(x) = (1-x)\sin[(x^2-1)^3]$$

Taking derivatives and evaluating them at $x = 1$ yields

$$\begin{aligned}
f'(x) &= 6x\left(-x+1\right)\left(x^2-1\right)^2\cos\left(\left(x^2-1\right)^3\right) - \sin\left(\left(x^2-1\right)^3\right)\\
f'(1) &= 0\\
f''(x) &= 36x^{11}\sin\left(x^6-3x^4+3x^2-1\right) - 36x^{10}\sin\left(x^6-3x^4+3x^2-1\right)\\
&\quad - 144x^9\sin\left(x^6-3x^4+3x^2-1\right) + 144x^8\sin\left(x^6-3x^4+3x^2-1\right)\\
&\quad + 216x^7\sin\left(x^6-3x^4+3x^2-1\right) - 216x^6\sin\left(x^6-3x^4+3x^2-1\right)\\
&\quad - 144x^5\sin\left(x^6-3x^4+3x^2-1\right) - 42x^5\cos\left(x^6-3x^4+3x^2-1\right)\\
&\quad + 144x^4\sin\left(x^6-3x^4+3x^2-1\right) + 30x^4\cos\left(x^6-3x^4+3x^2-1\right)\\
&\quad + 36x^3\sin\left(x^6-3x^4+3x^2-1\right) + 60x^3\cos\left(x^6-3x^4+3x^2-1\right)\\
&\quad - 36x^2\sin\left(x^6-3x^4+3x^2-1\right) - 36x^2\cos\left(x^6-3x^4+3x^2-1\right)\\
&\quad - 18x\cos\left(x^6-3x^4+3x^2-1\right) + 6\cos\left(x^6-3x^4+3x^2-1\right)\\
f^{(4)}(1) &= -192
\end{aligned}$$

Since n is even and $f^{(n)}(x) < 0$, $f(x)$ has a local maximum at $x = 1$. *NOTE: the higher order derivatives are too large to write so they aren't reported here. See code in appendix for all of the derivatives.*

Problem 2(b)

Find all critical points of the function and classify them as local maxima, local minima, or horizontal inflection points.

$$f(x) = \frac{1}{x^2-4x+5}, \quad -\infty < x < \infty$$

The local maxima and minima can be found by finding $f'(x)$, setting it equal to zero and solving for x. Doing so yields a critical point at $x = 2$. Using the test from the previous problem to determine the type of the critical point yields

$$f''(2) = -2$$

Therefore, this point is a local max.

The inflection points can be found by finding $f''(x)$, setting it equal to zero and solving for x. Doing so yields inflection points at

$$x = -\frac{\sqrt{3}}{3} + 2,\ \frac{\sqrt{3}}{3} + 2$$

So, there are three critical points, one is a local max, and two inflection points.

Problem 2(g)

$$f(x) = x^2 e^{-x}, \quad -\infty < x < \infty$$

Applying the same approach in this problem yields a local minimum at $x = 0$, a local max found at $x = 2$, a horizontal inflection point at $x = -\sqrt{2} + 2$, and another horizontal inflection point at $x = \sqrt{2} + 2$. So, four critical points were found for this problem.

Exercises 13.8

Problem 1(b)

Apply the Leibniz rule:

$$\frac{d}{dt}\int_3^t x^t \sin(x)\, dx$$

Leibniz Rule The order of differentiation and integration can be interchanged as follows:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\, dx + b'(t) f(b(t),t) - a'(t) f(a(t),t) \tag{7}$$

In this problem

$$\begin{aligned} a(t) &= 3 \\ a'(t) &= 0 \\ b(t) &= t \\ b'(t) &= 1 \end{aligned}$$

Also,

$$\frac{\partial}{\partial t}\left(x^t \sin(x)\right) = \frac{\partial}{\partial t}\left(e^{\ln(x^t)} \sin(x)\right) = \sin(x) \frac{\partial}{\partial t}\left(e^{t\ln(x)}\right) = x^t \ln(x) \sin(x)$$

So, inserting everything into Leibniz rule and simplifying yields

$$\boxed{\frac{d}{dt}\int_3^t x^t \sin(x)\, dx = \int_3^t x^t \ln(x)\sin(x)\, dx + t^t \sin(t)}$$

Problem 1(d)

$$\frac{d}{d\alpha}\int_{-2\alpha^2}^{-\alpha} e^{\alpha x^3}\, dx$$

For this problem

$$\begin{aligned} a(\alpha) &= -2\alpha^2 \\ a'(\alpha) &= -4\alpha \\ b(\alpha) &= -\alpha \\ b'(\alpha) &= -1 \end{aligned}$$

Also,

$$\frac{\partial f}{\partial \alpha} = x^3 e^{\alpha x^3}$$

Using the above derivatives in the definition of Leibniz rule and simplifying yields

$$\boxed{\frac{d}{d\alpha}\int_{-2\alpha^2}^{-\alpha} e^{\alpha x^3}\, dx = \int_{-2\alpha^3}^{-\alpha} x^3 e^{\alpha x^3}\, dx - e^{-\alpha^4} + 4\alpha e^{-8\alpha^6}}$$

Problem 1(f)

$$\frac{d}{dy}\int_{y^2}^{1}\frac{x}{x^3+y^3}\,dx$$

For this problem

$$\begin{aligned} a(y) &= y^2 \\ a'(y) &= 2y \\ b(y) &= 1 \\ b'(y) &= 0 \end{aligned}$$

and

$$\frac{\partial f}{\partial y} = \frac{-3xy^2}{x^3+y^3}$$

Inserting these into the definition of Leibniz rule,

$$\boxed{\frac{d}{dy}\int_{y^2}^{1}\frac{x}{x^3+y^3}\,dx = \int_{y^2}^{1}\frac{-3xy^2}{x^3+y^3}dx - \frac{2}{y^3+1}}$$

Problem 1(g)

$$\frac{d^2}{da^2}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv$$

This problem can be solved by using the Leibniz rule twice. That is,

$$\frac{d^2}{da^2}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv = \frac{d}{da}\left[\frac{d}{da}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv\right]$$

Applying Leibniz rule to the part inside the square brackets yields

$$\frac{d^2}{da^2}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv = \frac{d}{da}\left[-\int_{5a}^{a^2}2a\sin(v^2+a^2)\,dv + 2a\cos(a^4+a^2) - 5\cos(26a^2)\right]$$

Which can be written as

$$\begin{aligned}\frac{d^2}{da^2}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv = -\frac{d}{da}\int_{5a}^{a^2}2a\sin(v^2+a^2)\,dv - 4a^2\left(2a^2+1\right)\sin\left(a^4+a^2\right) \\ + 2\cos\left(a^4+a^2\right) + 260a\sin\left(26a^2\right)\end{aligned}$$

Applying Leibniz rule again,

$$\begin{aligned}\frac{d^2}{da^2}\int_{5a}^{a^2}\cos(v^2+a^2)\,dv = -\int_{5a}^{a^2}\left(4a^2\cos(v^2+a^2)+2\sin(v^2+a^2)\right)\,dv - 4a^2\sin(a^4+a^2) \\ + 270a\sin(26a^2) - 4a^2(2a^2+1)\sin(a^4+a^2) + 2\cos(a^4+a^2)\end{aligned}$$

Problem 2(b)

Derive the Taylor series of the given function $f(x)$ about $x = 0$, up to and including terms of second order, using the Leibniz rule to obtain $f'(x)$ and $f''(x)$.

$$f(x) = \int_{-x}^{\cos(x)} \frac{1}{t^3+1}\, dt$$

Evaluating the integral at $x = 0$,

$$f(0) = \int_0^1 \frac{1}{t^3+1} dt = \frac{1}{3}\log(2) + \frac{\sqrt{3}\pi}{9}$$

The equation for a Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Through second order terms, about $x = 0$,

$$f(x)|_0 = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ \ldots$$

Next, Leibniz rule is applied in the same way as it was in the preceding problems. Doing this and inserting the results into the equation for a Taylor series yields

$$\boxed{\int_{-x}^{\cos(x)} \frac{1}{t^3+1}\, dt = \frac{\sqrt{3}\pi}{9} + \frac{1}{3}\log(2) + x - \frac{x^2}{4} + \mathcal{O}\left(x^3\right)}$$

Problem 2(d)

$$f(x) = \int_0^{1+2x} e^{-xt^2}\, dt$$

This problem is solved in the same manner as the preceding problem. Doing so yields

$$\boxed{f(x) = \int_0^{1+2x} e^{-xt^2}\, dt = 1 + \frac{5x}{3} - \frac{19x^2}{10} + \mathcal{O}\left(x^3\right)}$$

Problem 2(f)

$$f(x) = \int_{-x}^{3\sin(x)} \cos(xt^2)\, dt$$

Applying the same process yields

$$\boxed{f(x) = \int_{-x}^{3\sin(x)} \cos(xt^2)\, dt = \frac{x\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} + \mathcal{O}\left(x^3\right)}$$

Problem 3

Show, by repeated differentiation of the formula

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$$

that

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

for $n = 0, 1, 2, 3, \ldots$.

This problem can be solved using integration by parts.

$$\int u\,dv = uv - \int v\,du$$

Letting

$$\mathcal{I}(n) = \int_0^\infty x^n e^{-x}\,dx$$

Now, $\mathcal{I}(0)$ is

$$\mathcal{I}(0) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$$

And,

$$\mathcal{I}(1) = \int_0^\infty xe^{-x}\,dx$$

Letting $u = x$ and $dv = e^{-x}dx$,

$$du = dx$$
$$v = -e^{-x}$$

Thus,

$$\mathcal{I}(1) = \int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx$$

Evaluating the above integral yields

$$\mathcal{I}(1) = 1\mathcal{I}(0)$$

Now

$$\mathcal{I}(2) = \int_0^\infty x^2 e^{-x}\,dx$$

Let $u = x^2$ and $dv = e^{-x}dx$,

$$du = 2x\,dx$$
$$v = -e^{-x}$$

Inserting these into integration by parts

$$\int_0^\infty x^2 e^{-x}\,dx = -2xe^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}\,dx = 2$$

So

$$\mathcal{I}(2) = 2\mathcal{I}(1)$$

Now evaluating $\mathcal{I}(n-1)$,

$$\mathcal{I}(n-1) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

Letting $u = x^{n-1}$ and $dv = e^{-x}\,dx$,

$$\begin{aligned} du &= (n-1)x^{n-2}\,dx \\ v &= -e^{-x} \end{aligned}$$

Inserting these into integration by parts

$$\mathcal{I}(n-1) = \int_0^\infty x^{n-1}e^{-x}\,dx = -x^{n-1}e^{-x}\Big|_0^\infty + \int_0^\infty (n-1)x^{n-2}e^{-x}\,dx$$

Next,

$$\mathcal{I}(n) = \int_0^\infty x^n e^{-x}\,dx$$

Letting $u = x^n$ and $dv = e^{-x}\,dx$,

$$\begin{aligned} du &= nx^{n-1}\,dx \\ v &= -e^{-x} \end{aligned}$$

Inserting these into integration by parts,

$$\begin{aligned} \mathcal{I}(n) = \int_0^\infty x^n e^{-x}\,dx &= -x^n e^{-x}\Big|_0^\infty + \int_0^\infty nx^{n-1}e^{-x}\,dx \\ &= n\int_0^\infty x^{n-1}e^{-x}\,dx = n\mathcal{I}(n-1) \end{aligned}$$

This recursion implies the following

$$\boxed{\mathcal{I}(n) = \int_0^\infty x^n e^{-x}\,dx = n!\,\mathcal{I}(0) = n!}$$

IPython SymPy Code

In [1]: %pylab inline

Populating the interactive namespace from numpy and matplotlib

In [2]: from sympy import *

In [3]: init_printing()

In [4]: from IPython.display import display

In [5]: x = Symbol('x')

In [6]:
```
def f(x):
    return 1/(2+x**(10))
```

In [7]: diff(f(x),x,20).expand().subs(x,0)

Out[7]:

$$304112751022080000$$

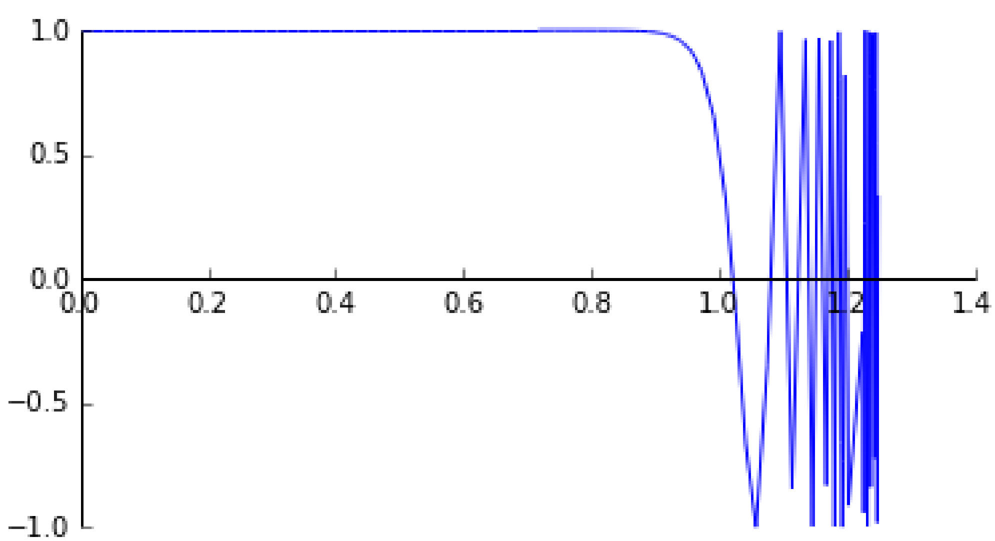
In [8]: series(f(x), x0=0, n=40)

Out[8]:

$$\frac{1}{2} - \frac{x^{10}}{4} + \frac{x^{20}}{8} - \frac{x^{30}}{16} + \mathcal{O}\left(x^{40}\right)$$

In [9]:
```
def f(x):
    return cos(x**(20))
```

In [10]: plot(f(x), (x,0,1.25), ylim=[-1.2,1.2])



Out[10]: <sympy.plotting.plot.Plot at 0x7f6b5a472dd8>

```
In [11]: for n in range(0,40):
             print("{} {}".format(n+1, diff(f(x), x, n+1).subs(x, 0)))
```

```
1 0
2 0
3 0
4 0
5 0
6 0
7 0
8 0
9 0
10 0
11 0
12 0
13 0
14 0
15 0
16 0
17 0
18 0
19 0
20 0
21 0
22 0
23 0
24 0
25 0
26 0
27 0
28 0
29 0
30 0
31 0
32 0
33 0
34 0
35 0
36 0
37 0
38 0
39 0
40 -4079576416239488671728056347980579471360000000000
```

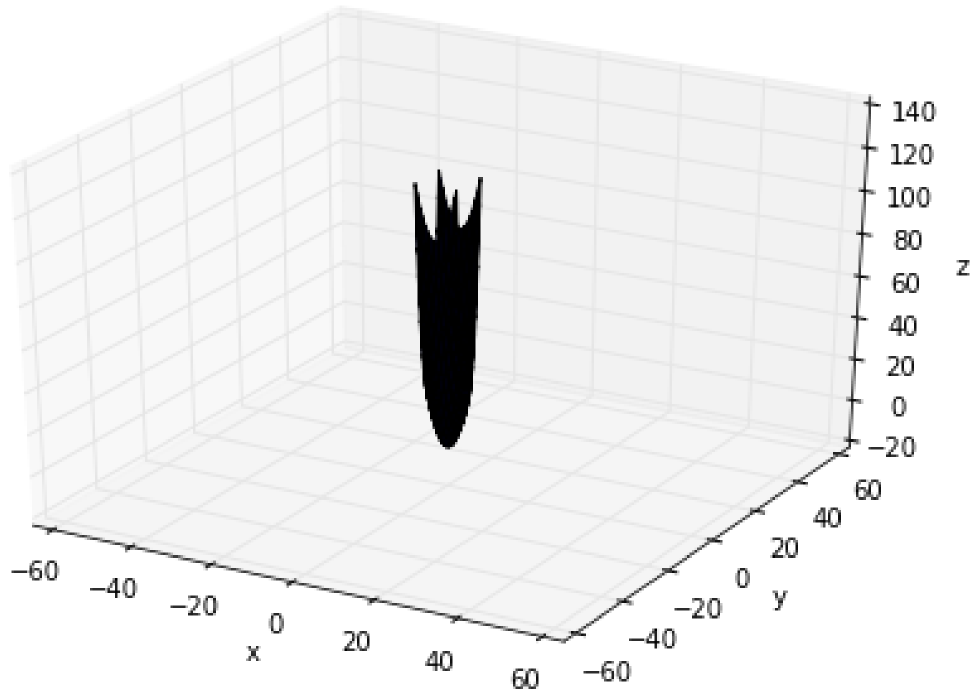
1 series(f(x), n=138)

2 Exercises 13.6

```
In [12]: y = Symbol("y")
```

```
In [13]: def f(x, y):
             return x**2+4*y**2-4
```

```
In [14]: mpmath.splot(f,[-5,5],[-5,5])
```



In [15]: `diff(f(x, y), x)`

Out[15]:

$$2x$$

In [16]: `diff(f(x, y), y)`

Out[16]:

$$8y$$

In [17]: `diff(f(x, y), x, y)`

Out[17]:

$$0$$

In [18]: `simplify((-32*x**2 - 128*y**2)/(512*y**3))`

Out[18]:

$$-\frac{x^2 + 4y^2}{16y^3}$$

In [19]: `simplify(Rational(-1/2*64/512))`

Out[19]:

$$-\frac{1}{16}$$

In [20]:
```
def f(x, y):
    return (x*(cos(pi*y)+1) + (x**3 + 8)*y)
```

In [21]:
```
diff(f(x,y),y)
```

Out[21]:

$$x^3 - \pi x \sin(\pi y) + 8$$

In [22]:
```
_.subs(x,-2).subs(y,1)
```

Out[22]:

$$0$$

In [23]:
```
def f(x, y):
    return x - y + sin(y)
```

In [24]:
```
diff(f(x, y), y)
```

Out[24]:

$$\cos(y) - 1$$

In [25]:
```
_.subs(y,0)
```

Out[25]:

$$0$$

In [26]:
```
def f(x,y):
    return x*y - y**3 - 1
```

In [27]:
```
fx = diff(f(x,y),x,1)
```

In [28]:
```
fxx = diff(f(x,y),x,2)
```

In [29]:
```
fy = diff(f(x,y),y,1)
```

In [30]:
```
fyy = diff(f(x,y),y,2)
```

In [31]:
```
fxy = diff(f(x,y),x,y)
```

In [32]:
```
simplify(-fx/fy)
```

Out[32]:

$$-\frac{y}{x - 3y^2}$$

In [33]:
```
simplify((2*fx*fy*fxy - fx**2*fyy - fy**2*fxx)/(fy**3))
```

Out[33]:

$$\frac{2xy}{(x - 3y^2)^3}$$

In [34]:
```
pprint(fx)
```

```
y
```

```
In [35]: y = Function('y')(x)

In [36]: dydx = solve(diff(x*y - y**3-1, x, 1), diff(y,x,1))[0]

In [37]: simplify(solve(diff(x*y - y**3-1, x, 2), diff(y,x,2))[0].subs(diff(y,x,1),dydx))
```

Out[37]:

$$\frac{2xy(x)}{\left(x - 3y^2(x)\right)^3}$$

```
In [38]: y = Symbol('y')

In [39]: def f(x,y):
             return y*cos(y) - x**3

In [40]: fx = diff(f(x,y),x,1)

In [41]: fx
```

Out[41]:

$$-3x^2$$

```
In [42]: fxx = diff(f(x,y),x,2)

In [43]: fxx
```

Out[43]:

$$-6x$$

```
In [44]: fy = diff(f(x,y),y,1)

In [45]: fy
```

Out[45]:

$$-y\sin(y) + \cos(y)$$

```
In [46]: fyy = diff(f(x,y),y,2)

In [47]: fyy
```

Out[47]:

$$-y\cos(y) + 2\sin(y)$$

```
In [48]: fxy = diff(f(x,y),x,y)

In [49]: fxy
```

Out[49]:

$$0$$

```
In [50]: simplify(-fx/fy)
```

Out[50]:

$$-\frac{3x^{2}}{y\sin{\left(y \right)} - \cos{\left(y \right)}}$$

```
In [51]: simplify((2*fx*fy*fxy - fx**2*fyy - fy**2*fxx)/(fy**3))
```

Out[51]:

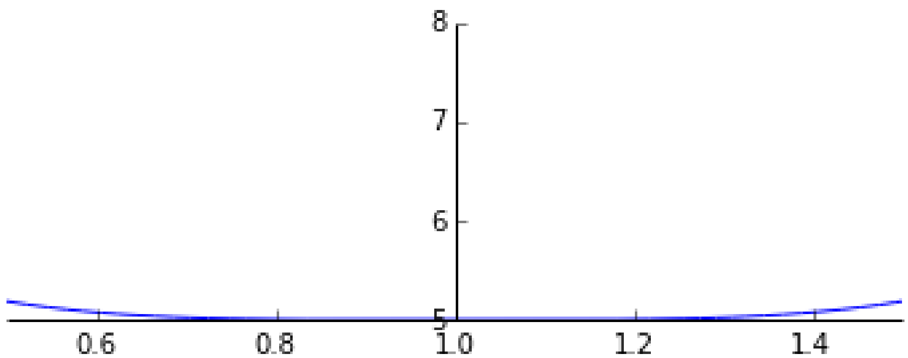
$$-\frac{3x}{\left(y\sin{\left(y \right)} - \cos{\left(y \right)}\right)^{3}} \left(3x^{3} \left(y\cos{\left(y \right)} + 2\sin{\left(y \right)}\right) + 2\left(y\sin{\left(y \right)} - \cos{\left(y \right)}\right)^{2}\right)$$

3 Exercises 13.7

```
In [52]: ff = Function('f')(x)
```

```
In [53]: ff = 3*(x-1)**4 + 5
```

```
In [54]: plot(ff, xlim=[0.5,1.5], ylim=[2,8])
```



Out[54]: <sympy.plotting.plot.Plot at 0x7f6b59168d68>

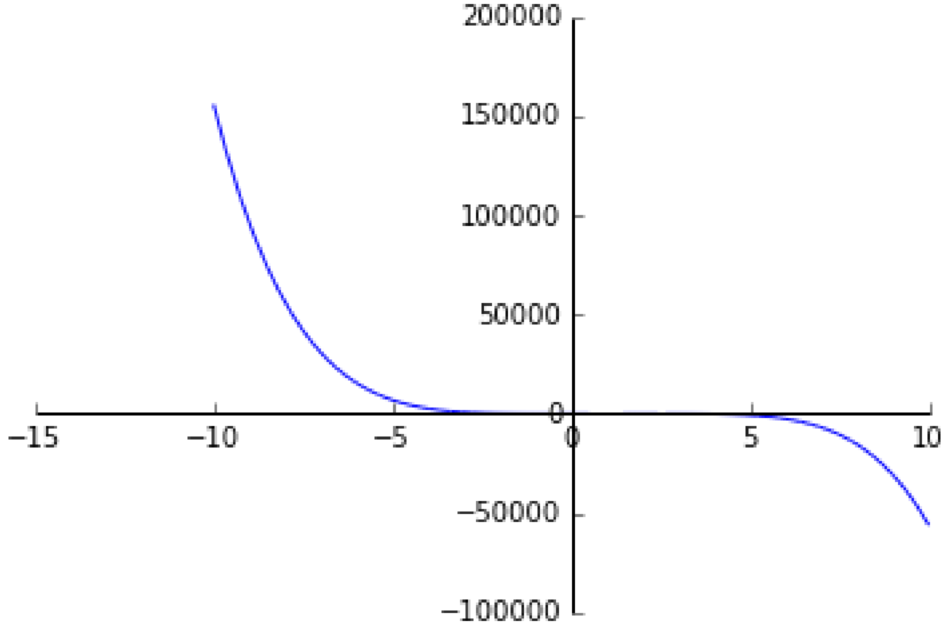
```
In [55]: diff(ff,x).expand()
```

Out[55]:

$$12x^{3} - 36x^{2} + 36x - 12$$

```
In [56]: ff = (x+1)*(x-3)*(1-x)**3
```

```
In [57]: plot(ff)
```



Out[57]: <sympy.plotting.plot.Plot at 0x7f6b591686a0>

```
In [58]: simplify(diff(ff,x,1))
```

Out[58]:

$$-5x^4 + 20x^3 - 18x^2 - 4x + 7$$

```
In [59]: _.subs(x,1)
```

Out[59]:

$$0$$

```
In [60]: simplify(diff(ff,x,2))
```

Out[60]:

$$-20x^3 + 60x^2 - 36x - 4$$

```
In [61]: _.subs(x,1)
```

Out[61]:

$$0$$

```
In [62]: simplify(diff(ff,x,3))
```

Out[62]:

$$-60x^2 + 120x - 36$$

In [63]:
```python
_.subs(x,1)
```

Out[63]:

$$24$$

In [64]:
```python
ff = exp(8*(x-1)**5)
```

In [65]:
```python
diff(ff,x,1).simplify()
```

Out[65]:

$$40\left(x-1\right)^{4}e^{8(x-1)^{5}}$$

In [66]:
```python
def classify_pt(inp_func, x0, maxIter=10, exp_choice=False):
    c = 1
    status=False
    pt_type = None
    while c < maxIter:
        fpx = diff(ff,x,c)
        fpx0 = fpx.subs(x,x0).simplify()
        print("\n----------------------------------------")
        if c == 1:
            print("{}st order derivative:".format(c))
        elif c == 2:
            print("{}nd order derivative:".format(c))
        elif c==3:
            print("{}rd order derivative:".format(c))
        else:
            print("{}th order derivative:".format(c))
        print("----------------------------------------")
        if exp_choice:
            display(expand(fpx))
        else:
            display(simplify(fpx))
        display(simplify(fpx0))
        if fpx0 != 0 and c >= 2:
            if c%2==0 and fpx0 < 0:
                print("Local maximum found at x = {}".format(x0))
                pt_type="Local max"
            elif c%2==0 and fpx0 > 0:
                print("Local minimum found at x = {}".format(x0))
                pt_type="Local min"
            elif c%2!=0:
                print("Horizontal inflection point found at x = {}.".format(x0))
                pt_type="Inflection point"
            else:
                print("Error: unable to classify.")
            status=True
            break
        c+=1
    return pt_type
```

In [67]:
```python
classify_pt(ff,1)
```

```
----------------------------------------
1st order derivative:
----------------------------------------
```

$$40\left(x-1\right)^{4}e^{8\left(x-1\right)^{5}}$$

$$0$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\left(x-1\right)^{3}\left(1600\left(x-1\right)^{5}+160\right)e^{8\left(x-1\right)^{5}}$$

$$0$$

```
----------------------------------------
3rd order derivative:
----------------------------------------
```

$$\left(x-1\right)^{2}\left(64000\left(x-1\right)^{10}+19200\left(x-1\right)^{5}+480\right)e^{8\left(x-1\right)^{5}}$$

$$0$$

```
----------------------------------------
4th order derivative:
----------------------------------------
```

$$320\left(x-1\right)\left(8000\left(x-1\right)^{15}+4800\left(x-1\right)^{10}+480\left(x-1\right)^{5}+3\right)e^{8\left(x-1\right)^{5}}$$

$$0$$

```
----------------------------------------
5th order derivative:
----------------------------------------
```

$$\left(102400000\left(x-1\right)^{20}+102400000\left(x-1\right)^{15}+23040000\left(x-1\right)^{10}+960000\left(x-1\right)^{5}+960\right)e^{8\left(x-1\right)^{5}}$$

$$960$$

```
Horizontal inflection point found at x = 1.
```

Out[67]: 'Inflection point'

```
In [68]: ff = (1-x)*sin((x**2-1)**3)
```

```
In [69]: classify_pt(ff,1,exp_choice=True)
```

```
----------------------------------------
1st order derivative:
----------------------------------------
```

$$-6x^6\cos\left(x^6-3x^4+3x^2-1\right)+6x^5\cos\left(x^6-3x^4+3x^2-1\right)+12x^4\cos\left(x^6-3x^4+3x^2-1\right)-12x^3\cos\left(x^6-3x^4+3x^2-1\right)-$$

$$0$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$36x^{11}\sin\left(x^6-3x^4+3x^2-1\right)-36x^{10}\sin\left(x^6-3x^4+3x^2-1\right)-144x^9\sin\left(x^6-3x^4+3x^2-1\right)+144x^8\sin\left(x^6-3x^4+3x^2-\right.$$

$$0$$

```
----------------------------------------
3rd order derivative:
----------------------------------------
```

$$216x^{16}\cos\left(x^6-3x^4+3x^2-1\right)-216x^{15}\cos\left(x^6-3x^4+3x^2-1\right)-1296x^{14}\cos\left(x^6-3x^4+3x^2-1\right)+1296x^{13}\cos\left(x^6-3x^4+\right.$$

$$0$$

```
----------------------------------------
4th order derivative:
----------------------------------------
```

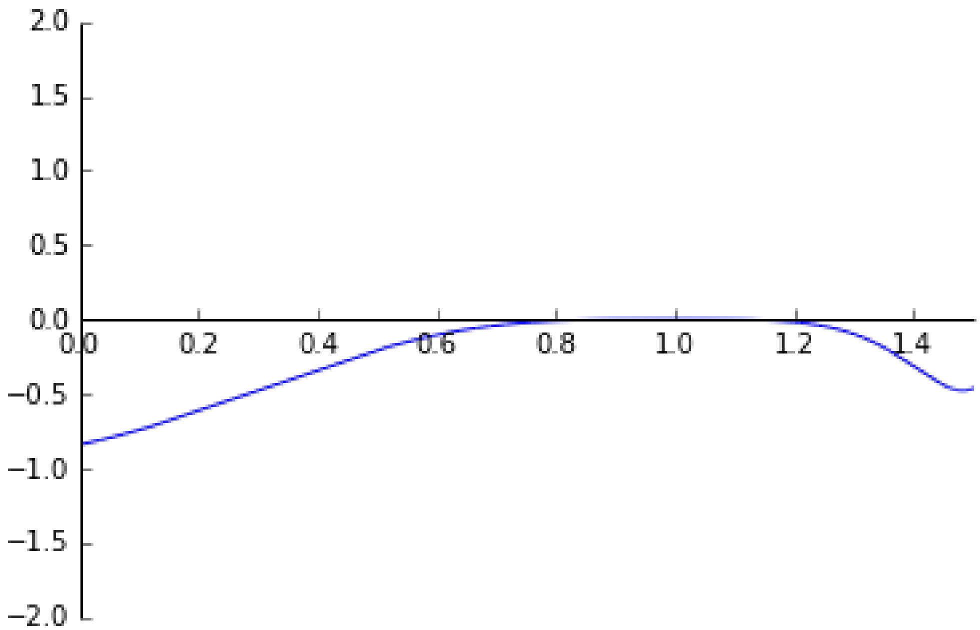
$$-1296x^{21}\sin\left(x^6-3x^4+3x^2-1\right)+1296x^{20}\sin\left(x^6-3x^4+3x^2-1\right)+10368x^{19}\sin\left(x^6-3x^4+3x^2-1\right)-10368x^{18}\sin\left(x^6-\right.$$

$$-192$$

```
Local maximum found at x = 1
```

Out[69]: 'Local max'

In [70]:
```
plot(ff,xlim=[0,1.5],ylim=[-2,2])
```



Out[70]: <sympy.plotting.plot.Plot at 0x7f6b590609e8>

In [71]:
```
ff = 1/(x**2 - 4*x + 5)
```

In [72]:
```
def find_crit_pts(input_function):
    crit_pts = solve(diff(input_function,x),x)
    infl_pts = solve(diff(input_function,x,2),x)
    cpt = []
    pt_type = []
    pt_counter = 1
    for pt in crit_pts:
        print("\n========================================================")
        print("Point {} at x = {}".format(pt_counter, pt))
        print("========================================================\n")
        cpt.append(pt)
        pt_type.append(classify_pt(input_function, pt.simplify()))
        pt_counter += 1
    for pt in infl_pts:
        print("\n========================================================")
        print("Point {} at x = {}".format(pt_counter, pt))
        print("========================================================\n")
        cpt.append(pt)
        pt_type.append(classify_pt(input_function, pt.simplify()))
        pt_counter += 1
    return (cpt, pt_type)
```

In [73]:
```
find_crit_pts(ff)
```

```
========================================================
Point 1 at x = 2
========================================================
```

```
----------------------------------------
1st order derivative:
----------------------------------------
```

$$\frac{-2x+4}{\left(x^2-4x+5\right)^2}$$

$$0$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\frac{1}{\left(x^2-4x+5\right)^3}\left(-2x^2+8x+8\left(x-2\right)^2-10\right)$$

$$-2$$

```
Local maximum found at x = 2

========================================================
Point 2 at x = -sqrt(3)/3 + 2
========================================================

----------------------------------------
1st order derivative:
----------------------------------------
```

$$\frac{-2x+4}{\left(x^2-4x+5\right)^2}$$

$$\frac{3\sqrt{3}}{8}$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\frac{1}{\left(x^2-4x+5\right)^3}\left(-2x^2+8x+8\left(x-2\right)^2-10\right)$$

$$0$$

```
----------------------------------------
3rd order derivative:
----------------------------------------
```

$$\frac{24}{\left(x^2-4x+5\right)^4}\left(x-2\right)\left(x^2-4x-2\left(x-2\right)^2+5\right)$$

$$-\frac{27\sqrt{3}}{16}$$

```
Horizontal inflection point found at x = -sqrt(3)/3 + 2.

============================================================
Point 3 at x = sqrt(3)/3 + 2
============================================================


-------------------------------------------
1st order derivative:
-------------------------------------------
```

$$\frac{-2x+4}{\left(x^{2}-4x+5\right)^{2}}$$

$$-\frac{3\sqrt{3}}{8}$$

```
-------------------------------------------
2nd order derivative:
-------------------------------------------
```

$$\frac{1}{\left(x^{2}-4x+5\right)^{3}}\left(-2x^{2}+8x+8\left(x-2\right)^{2}-10\right)$$

$$0$$

```
-------------------------------------------
3rd order derivative:
-------------------------------------------
```

$$\frac{24}{\left(x^{2}-4x+5\right)^{4}}\left(x-2\right)\left(x^{2}-4x-2\left(x-2\right)^{2}+5\right)$$

$$\frac{27\sqrt{3}}{16}$$

```
Horizontal inflection point found at x = sqrt(3)/3 + 2.
```

```
Out[73]: ([2, -sqrt(3)/3 + 2, sqrt(3)/3 + 2],
          ['Local max', 'Inflection point', 'Inflection point'])
```

```
In [74]: ff = x**2*exp(-x)
```

```
In [75]: find_crit_pts(ff)
```

```
============================================================
Point 1 at x = 0
============================================================


-------------------------------------------
1st order derivative:
-------------------------------------------
```

$$x(-x+2)e^{-x}$$

$$0$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\left(x^2-4x+2\right)e^{-x}$$

$$2$$

```
Local minimum found at x = 0

========================================================
Point 2 at x = 2
========================================================


----------------------------------------
1st order derivative:
----------------------------------------
```

$$x(-x+2)e^{-x}$$

$$0$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\left(x^2-4x+2\right)e^{-x}$$

$$-\frac{2}{e^2}$$

```
Local maximum found at x = 2

========================================================
Point 3 at x = -sqrt(2) + 2
========================================================


----------------------------------------
1st order derivative:
----------------------------------------
```

$$x(-x+2)e^{-x}$$

$$\frac{-2+2\sqrt{2}}{e^{-\sqrt{2}+2}}$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\left(x^{2} - 4x + 2\right) e^{-x}$$

$$0$$

```
----------------------------------------
3rd order derivative:
----------------------------------------
```

$$\left(-x^{2} + 6x - 6\right) e^{-x}$$

$$-\frac{2\sqrt{2}}{e^{-\sqrt{2}+2}}$$

```
Horizontal inflection point found at x = -sqrt(2) + 2.

========================================================
Point 4 at x = sqrt(2) + 2
========================================================


----------------------------------------
1st order derivative:
----------------------------------------
```

$$x\left(-x + 2\right) e^{-x}$$

$$-\frac{2 + 2\sqrt{2}}{e^{\sqrt{2}+2}}$$

```
----------------------------------------
2nd order derivative:
----------------------------------------
```

$$\left(x^{2} - 4x + 2\right) e^{-x}$$

$$0$$

```
----------------------------------------
3rd order derivative:
----------------------------------------
```

$$\left(-x^{2} + 6x - 6\right) e^{-x}$$

$$\frac{2\sqrt{2}}{e^{\sqrt{2}+2}}$$

```
Horizontal inflection point found at x = sqrt(2) + 2.
```

```
Out[75]: ([0, 2, -sqrt(2) + 2, sqrt(2) + 2],
          ['Local min', 'Local max', 'Inflection point', 'Inflection point'])
```

4 Exercises 13.8

```
In [76]: t = Symbol("t")
```

```
In [77]: alpha = Symbol("alpha")
```

```
In [78]: diff(x**t*sin(x),t)
```

Out[78]:

$$x^t \log(x) \sin(x)$$

```
In [79]: exp((-2*alpha**2)**3).simplify()
```

Out[79]:

$$e^{-8\alpha^6}$$

```
In [80]: diff(x/(x**3 + y**3),y)
```

Out[80]:

$$-\frac{3xy^2}{(x^3+y^3)^2}$$

```
In [81]: simplify(-2*y*y**2/((y**2)**3+y**3))
```

Out[81]:

$$-\frac{2}{y^3+1}$$

```
In [82]: a = Symbol("a")
```

```
In [83]: diff(-5*cos(26*a**2),a)
```

Out[83]:

$$260a\sin\left(26a^2\right)$$

```
In [84]: part2b = integrate(1/(t**3+1),(t,-x,cos(x))).simplify()
```

```
In [85]: part2b.subs(x,0)
```

Out[85]:

$$\frac{1}{3}\log(2) + \frac{\sqrt{3}\pi}{9}$$

```
In [86]: help(series)
```

```
Help on function series in module sympy.series.series:

series(expr, x=None, x0=0, n=6, dir='+')
    Series expansion of expr around point `x = x0`.

    See the doctring of Expr.series() for complete details of this wrapper.
```

```
In [87]: series(part2b, n=3)
```

Out[87]:

$$\frac{\sqrt{3}\pi}{9} + \frac{1}{3}\log\left(2\right) + x - \frac{x^{2}}{4} + \mathcal{O}\left(x^{3}\right)$$

```
In [88]: part2d = integrate(exp(-x*t**2),(t,0,1+2*x))
```

```
In [89]: part2d.simplify().subs(x,0)
```

Out[89]:

$$1$$

```
In [90]: series(part2d,n=3)
```

Out[90]:

$$1 + \frac{5x}{3} - \frac{19x^{2}}{10} + \mathcal{O}\left(x^{3}\right)$$

```
In [91]: part2f = integrate(cos(x*t**2),(t,-x,3*sin(x))).simplify()
```

```
In [92]: part2f
```

Out[92]:

$$\frac{\sqrt{2}\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{8\sqrt{x}\Gamma\left(\frac{5}{4}\right)}\left(C\left(\frac{\sqrt{2}x^{\frac{3}{2}}}{\sqrt{\pi}}\right) + C\left(\frac{3\sqrt{2}}{\sqrt{\pi}}\sqrt{x}\sin\left(x\right)\right)\right)$$

```
In [93]: series(part2f,n=3)
```

Out[93]:

$$\frac{x\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} + \mathcal{O}\left(x^{3}\right)$$

```
In [94]: integrate(x**3*exp(-x),(x,0,oo))
```

Out[94]:

$$6$$

```
In [107]: integrate(exp(-a*x),x)
```

Out[107]:

$$\begin{cases} x & \text{for}\: a = 0 \\ - \frac{1}{a} e^{- a x} & \text{otherwise} \end{cases}$$

```
In [111]: diff(exp(-a*x),x)
```

Out[111]:

$$- a e^{- a x}$$

```
In [112]: diff(_,a)
```

Out[112]:

$$axe^{-ax} - e^{-ax}$$

```
In [113]: diff(_,a)
```

Out[113]:

$$-ax^2e^{-ax} + 2xe^{-ax}$$

```
In [114]: diff(_,a)
```

Out[114]:

$$ax^3e^{-ax} - 3x^2e^{-ax}$$

```
In [ ]:
```