EE 5350 - Homework 5

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Problem 3.1

Problem Statement

Determine the z-transform, including the ROC, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n] - u[n-10])$

Solution

(a) $\left(\frac{1}{2}\right)^n u[n]$

$$\mathcal{Z}\{(1/2)^n u[n]\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

Expressing the sum in closed form yields

$$\boxed{\mathcal{Z}\{(1/2)^n u[n]\} = \frac{1}{1 - \frac{1}{2} z^{-1}}}$$

This closed form is valid if and only if

$$\left|\frac{1}{2} z^{-1}\right| < 1$$

Therefore, the region of convergence is

$$\boxed{|z| > \frac{1}{2}}$$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

$$\mathcal{Z}\{-(1/2)^n u[-n-1]\} = \sum_{n=-\infty}^{\infty} -\left(\frac{1}{2}\right)^n u[-n-1]z^{-n}$$

The unit step turns on if $-n-1 \geqslant 0$. So, the sum changes to

$$\mathcal{Z}\{-(1/2)^n u[-n-1]\} = \sum_{n=-\infty}^{-1} -\left(\frac{1}{2}\right)^n z^{-n}$$

Letting $n = -n$,

$$\mathcal{Z}\{-(1/2)^n u[-n-1]\} = -\sum_{n=1}^{\infty}(2z)^n$$

The sum can be rewritten as

$$-\sum_{n=1}^{\infty}(2z)^n = -\sum_{n=0}^{\infty}(2z)^n + 1$$

because the first term of the sequence is equal to one. Now, the closed form is

$$\mathcal{Z}\{-(1/2)^n u[-n-1]\} = -\frac{1}{1+2z} + 1$$

and is valid if and only if $|2z| < 1$. Simplifying the closed form yields

$$\boxed{\mathcal{Z}\{-(1/2)^n u[-n-1]\} = \frac{1}{1-\frac{1}{2}z^{-1}}}$$

And the region of convergence is

$$\boxed{|z| < \frac{1}{2}}$$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

The z-transform of this sequence is

$$\mathcal{Z}\{(1/2)^n u[-n]\} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[-n]z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}z^{-1}\right)^n$$

Letting $n = -n$,

$$\sum_{n=-\infty}^{0} \left(\frac{1}{2}z^{-1}\right)^n = \sum_{n=0}^{\infty} (2z)^n$$

The closed form of the sum is

$$\sum_{n=0}^{\infty} (2z)^n = \frac{1}{1-2z}$$

and is valid iff $|2z| < 1$. Multiplying the numerator and denominator by $1/2z^{-1}$ yields

$$\boxed{\mathcal{Z}\{(1/2)^n u[-n]\} = \frac{-\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}}$$

The region of convergence is

$$\boxed{|z| < \frac{1}{2}}$$

(d) $\delta[n]$

$$\boxed{\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = 1}$$

The region of convergence for this sequence is all z.

(e) $\delta[n-1]$

$$\boxed{\mathcal{Z}\{\delta[n-1]\} = \sum_{n=-\infty}^{\infty} \delta[n-1]z^{-n} = z^{-1}}$$

The region of convergence is all z except for $z = 0$.

(f) $\delta[n+1]$

$$\boxed{\mathcal{Z}\{\delta[n+1]\} = \sum_{n=-\infty}^{\infty} \delta[n+1]z^{-n} = z}$$

The region of convergence is

$$\boxed{|z| < \infty}$$

(g) $\left(\frac{1}{2}\right)^n (u[n] - u[n-10])$

$$\begin{aligned}\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right\} &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n (u[n]-u[n-10])z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}u[n] - \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}u[n-10]\end{aligned}$$

$$\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right\} = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n - \sum_{n=10}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

This sum can be rewritten as

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n - \sum_{n=10}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n = \cancel{\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n} - \cancel{\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n} + \sum_{n=0}^{9}\left(\frac{1}{2}z^{-1}\right)^n$$

Therefore,

$$\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right\} = \sum_{n=0}^{9}\left(\frac{1}{2}z^{-1}\right)^n$$

The following closed form is applied:

$$\sum_{n=N}^{M} x^n = \frac{x^N - x^{M+1}}{1-x} \quad \text{iff } x \neq 1$$

$$\boxed{\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right\} = \frac{1-\left(\frac{1}{2}z^{-1}\right)^{10}}{1-\frac{1}{2}z^{-1}}}$$

This closed form is valid iff

$$\frac{1}{2}z^{-1} \neq 1$$

Therefore,

$$\boxed{z \neq \frac{1}{2}}$$

Problem 3.4

Problem Statement

Consider the z-transform $X(z)$ whose pole-zero plot is as shown in Figure P3.4.

(a) Determine the ROC of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) How many possible two sided sequences have the pole-zero plot shown in Figure P3.4?

(c) Is it possible for the pole-zero plot in Figure P3.4 to be associated with a sequence that is both stable and causal? If so, give the appropriate ROC.

Solution

(a) Because it is known that the DTFT exists, the Region of Convergence must contain the unit circle. Also, the region of convergence cannot contain any poles. Therefore, the sequence must be two-sided and the ROC is given by

$$\boxed{\frac{1}{3} < |z| < 2}$$

(b) It is possible for two sequences to have the pole-zero plot shown in the figure. The ROCs for the two, two-sided sequences are $1/3 < |z| < 2$ and $2 < |z| < 3$.

(c) The pole-zero plot cannot be associated with a sequence that is both stable and causal because both possible ROCs represent two sided sequences. This means that both possible sequences are noncausal.

Problem 3.6

Problem Statement

Following are several z-transforms. For each, determine the inverse z-transform using both methods−partial fraction expansion and power series expansion−discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}$

(b) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \; |z| < \frac{1}{2}$

(c) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, \; |z| > \frac{1}{2}$

(d) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}, \; |z| > \frac{1}{2}$

(e) $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a}, \; |z| > \left|\frac{1}{a}\right|$

Solution

(a) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \; |z| > \frac{1}{2}$

There's no need to perform a partial fraction expansion. Because $|z| > 1/2$, the sequence must be right sided. Because the ROC contains the unit circle, the Fourier transform exists.

There's no need to perform a partial fraction expansion. By inspection,

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

Performing the power series expansion,

$$\frac{1}{1+\frac{1}{2}z^{-1}} = 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \ldots$$

Therefore,

$$\boxed{x[n] = \left(-\frac{1}{2}\right)^n u[n]}$$

(b) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}, \ |z| < \frac{1}{2}$

Because of the region of convergence, the sequence is left sided and the Fourier transform does not exist. By inspection,

$$x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

Performing the power series expansion yields

$$\frac{1}{1+\frac{1}{2}z^{-1}} = 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3} + \ldots$$

$$\boxed{x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]}$$

(c) $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}, \ |z| > \frac{1}{2}$

The ROC contains the unit circle and corresponds to a right sided sequence. Therefore, the Fourier transform exists.

The denominator can be factored as

$$1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)$$

Using a partial fraction expansion, the rational function can be expressed as

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{A_1}{1+\frac{1}{2}z^{-1}} + \frac{A_2}{1+\frac{1}{4}z^{-1}}$$

Multiplying both sides by $1 + 1/2z^{-1}$ and evaluating at $z^{-1} = -2$,

$$A_1 = 4$$

Doing the same for the second term,

$$A_2 = -3$$

Therefore,

$$X(z) = \frac{4}{1 + \frac{1}{2}z^{-1}} - \frac{3}{1 + \frac{1}{4}z^{-1}}$$

From this expression, it is recognized that

$$\mathcal{Z}\{a^n u[n]\} = \frac{1}{1 - az^{-1}}$$

Thus,

$$\boxed{x[n] = 4\left(-\frac{1}{2}\right)^n u[n] - 3\left(-\frac{1}{4}\right)^n u[n]}$$

Performing the power series expansion using long division yields

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = 1 - \frac{5}{4}z^{-1} + \frac{13}{16}z^{-2} + \dots$$

The first few terms of the power series expansion yield the same numbers as the result obtained using partial fraction expansion. However, it would be more difficult to determine the original sequence from the power series.

(d) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}$, $|z| > \frac{1}{2}$

In this case, the ROC contains the unit circle and corresponds to a right sided sequence. The Fourier transform exists.

Factoring the denominator yields

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1 - \frac{1}{2}z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

which simplifies to

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}$$

Therefore,

$$\boxed{x[n] = \left(-\frac{1}{2}\right)^n u[n]}$$

The power series expansion yields

$$\frac{1}{1 + \frac{1}{2}z^{-1}} = 1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \dots$$

The original sequence $x[n]$ can also be determined by inspecting the above equation.

$$x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

The unit step is added because the given ROC indicates a right sided sequence.

(e) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}$, $|z| > \left|\frac{1}{a}\right|$

If $|a| > 1$, the ROC includes the unit circle and the Fourier transform exists. The expression for $X(z)$ can be simplified.

$$X(z) = \frac{1}{z^{-1} - a} - \frac{az^{-1}}{z^{-1} - a}$$

Multiplying by $(-1/a)/(-1/a)$,

$$X(z) = \frac{-\frac{1}{a}}{1 - \frac{1}{a}z^{-1}} + \frac{z^{-1}}{1 - \frac{1}{a}z^{-1}}$$

Using the shift theorem,

$$x[n] = -\frac{1}{a}\left(\frac{1}{a}\right)^n u[n] + \left(\frac{1}{a}\right)^{n-1} u[n-1]$$

Which can be simplified as follows:

$$x[n] = -a^{-1}a^{-n}u[n] + a^{-(n-1)}u[n-1]$$

$$\boxed{x[n] = -a^{-(n+1)}u[n] + a^{-(n-1)}u[n-1]}$$

Long division yields a power series expansion of

$$\frac{1 - az^{-1}}{z^{-1} - a} = -\frac{1}{a} - \frac{1 - a^2}{a^2}z^{-1} - \frac{1 - a^2}{a^3}z^{-2} + \ldots$$

Again, it is difficult to see the original sequence from the power series expansion of the z-transform.

Problem 3.7

Problem Statement

The input to a causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

(b) What is the ROC of $Y(z)$?

(c) Determine $y[n]$.

Solution

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the ROC.

This can be accomplished by taking the z-transform of $x[n]$ and then computing the transfer function as $H(z) = Y(z)/X(z)$.

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} \left(u[-n-1] + \left(\frac{1}{2}\right)^n u[n] \right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[-n-1] z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n u[n]$$

The sums can be rewritten so that the unit steps are equal to one.

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{-1} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

The first summation can be rewritten as

$$\sum_{n=-\infty}^{-1} z^{-n} = \sum_{n=1}^{\infty} z^n = \sum_{n=0}^{\infty} (z^n) - 1$$

Therefore, the z-transform becomes

$$\mathcal{Z}\{x[n]\} = -1 + \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

Putting the sums in closed form yields

$$X(z) = -1 + \frac{1}{1-z} + \frac{1}{1 - \frac{1}{2} z^{-1}}$$

Manipulating further yields

$$X(z) = \frac{\frac{1}{2}}{(1-z)\left(1 - \frac{1}{2} z^{-1}\right)}$$

The ROC for $X(z)$ is $|z| < 1$ and $|1/2z^{-1}| < 1$. More generally,

$$\frac{1}{2} < |z| < 1$$

Using the definition of the transfer function,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{-\frac{1}{2} z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})}}{\frac{\frac{1}{2}}{(1-z)\left(1-\frac{1}{2}z^{-1}\right)}}$$

Which simplifies to

$$\boxed{H(z) = \frac{1 - z^{-1}}{1 + z^{-1}}}$$

The ROC of $H(z)$ is

$$\boxed{|z| > 1}$$

(b) What is the ROC of $Y(z)$?

The ROC of $Y(z)$ is

$$\boxed{|z| > 1}$$

(c) Determine $y[n]$.

This subpart can be solve by taking the inverse z-transform of $Y(z)$. First, a partial fraction expansion yields

$$Y(z) = \frac{-\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1 + z^{-1}}$$

Therefore,

$$\boxed{y[n] = \left[\frac{1}{3}(-1)^n - \frac{1}{3}\left(\frac{1}{2}\right)^n\right] u[n]}$$

Problem 3.11

Problem Statement

Following are four z-transforms. Determine which ones *could* be the z-transform of a *causal* sequence. Do not evaluate the inverse transform. You should be able to give the answer by inspection. Clearly state your reasons in each case.

(a) $\dfrac{(1 - z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)}$

(b) $\dfrac{(z - 1)^2}{z - \frac{1}{2}}$

(c) $\dfrac{\left(z - \frac{1}{4}\right)^5}{\left(z - \frac{1}{2}\right)^6}$

(d) $\dfrac{\left(z - \frac{1}{4}\right)^6}{\left(z - \frac{1}{2}\right)^5}$

Solution

(a) $\dfrac{(1-z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)}$

The highest power of z is z^{-1}. Therefore, the sequence could be causal.

(b) $\dfrac{(z-1)^2}{z-\frac{1}{2}}$

Multiplying the numerator and denominator by z^{-1}, the highest power of z in the numerator is still z. Therefore, the sequence cannot be causal because z to a positive power corresponds to future values in the time domain.

(c) $\dfrac{\left(z-\frac{1}{4}\right)^5}{\left(z-\frac{1}{2}\right)^6}$

Multiplying the numerator and denominator by z^{-6}, the highest power of z is z^{-1}. Therefore, the sequence could be causal.

(d) $\dfrac{\left(z-\frac{1}{4}\right)^6}{\left(z-\frac{1}{2}\right)^5}$

Multiplying the numerator and denominator by z^{-5}, the highest power of z is z^1. Therefore, the sequence can't be causal.

Problem 3.27

Problem Statement

Determine the unilateral z-transform, including the ROC for each of the following sequences:

(a) $\delta[n]$

(b) $\delta[n-1]$

(c) $\delta[n+1]$

(d) $\left(\dfrac{1}{2}\right)^n u[n]$

(e) $-\left(\dfrac{1}{2}\right)^n u[-n-1]$

(f) $\left(\dfrac{1}{2}\right)^n u[-n]$

(g) $\left\{\left(\dfrac{1}{2}\right)^n + \left(\dfrac{1}{4}\right)^n\right\} u[n]$

(h) $\left(\frac{1}{2}\right)^{n-1} u[n-1]$

Solution

(a) $\delta[n]$

The unilateral z-transform is given by

$$\boxed{\sum_{n=0}^{\infty} \delta[n] z^{-n} = 1}$$

The ROC is all z.

(b) $\delta[n-1]$

$$\boxed{\sum_{n=0}^{\infty} \delta[n-1] z^{-n} = z}$$

The ROC is all z.

(c) $\delta[n+1]$

$$\boxed{\sum_{n=0}^{\infty} \delta[n+1] z^{n} = 0}$$

The ROC is all z.

(d) $\left(\frac{1}{2}\right)^{n} u[n]$

In this case, the unit step has no effect.

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} u[n] z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}}$$

The ROC is

$$\boxed{|z| > \frac{1}{2}}$$

(e) $-\left(\frac{1}{2}\right)^{n} u[-n-1]$

In order for this signal to turn on, $n \leqslant -1$. Therefore,

$$\boxed{\sum_{n=0}^{\infty} -\left(\frac{1}{2}\right)^{n} u[-n-1] z^{-n} = 0}$$

The ROC is all z.

(f) $\left(\frac{1}{2}\right)^n u[-n]$

Because the sequence is left sided, the unilateral z-transform is only evaluated at $n = 0$.

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n u[-n] z^{-n} = 1}$$

The ROC is all z.

(g) $\left\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right\} u[n]$

$$\sum_{n=0}^{\infty} \left\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right\} u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{4}z^{-1}\right)^n$$

Expressing the sums in closed form yields

$$\boxed{\sum_{n=0}^{\infty} \left\{\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right\} u[n] z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}}}$$

The ROC is

$$\boxed{|z| > \frac{1}{2}}$$

(h) $\left(\frac{1}{2}\right)^{n-1} u[n-1]$

The unit step turns on for $n \geqslant 1$. Therefore, unilateral z-transform becomes

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} u[n-1] z^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} z^n$$

Changing variables such that $k = n - 1$, the sum becomes

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} z^n = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-(k+1)} = z^{-1} \sum_{k=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^k$$

The final result is

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} u[n-1] z^{-n} = \frac{z^{-1}}{1 - \frac{1}{2}z^{-1}}}$$

The ROC is

$$\boxed{|z| > 1}$$

Problem 3.34

Problem Statement

Determine a sequence $x[n]$ whose z-transform is $X(z) = e^z + e^{1/z}$, $z \neq 0$.

Solution

The Taylor series expansion for an exponential function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty$$

Therefore,

$$X(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

Letting $n = -n$ in the first sum,

$$X(z) = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

Therefore,

$$x[n] = \begin{cases} \frac{1}{(-n)!} & n < 0 \\ \frac{1}{n!} & n \geqslant 0 \end{cases}$$

Expressed in terms of unit steps,

$$\boxed{x[n] = \frac{1}{(-n)!} u[-n] + \frac{1}{n!} u[n]}$$

Problem 3.36

Problem Statement

For each of the following sequences, determine the z-transform $X(z)$ and ROC, and sketch the pole-zero diagram:

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]$, $|a| < |b| < |c|$

(b) $x[n] = n^2 a^n u[n]$

(c) $x[n] = e^{n^4} \cos\left(\frac{\pi}{12} n\right) u[n] - e^{n^4} \cos\left(\frac{\pi}{12} n\right) u[n-1]$

Solution

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]$, $|a| < |b| < |c|$

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} (a^n u[n] + b^n u[n] + c^n u[-n-1]) z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (bz^{-1})^n + \sum_{n=-\infty}^{-1} (cz^{-1})^n$$

Letting $n = -n$, the third sum becomes

$$\sum_{n=-\infty}^{-1} (cz^{-1})^n = \sum_{n=1}^{\infty} (c^{-1}z)^n = \sum_{n=0}^{\infty} (c^{-1}z)^n - 1$$

Therefore, the z-transform is

$$\boxed{\mathcal{Z}\{x[n]\} = \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} + \frac{1}{1-c^{-1}z} - 1}$$

The ROC is

$$\boxed{|b| < |z| < |c|}$$

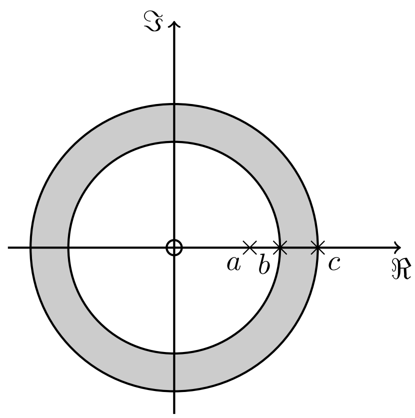

Figure 1: Pole-zero plot for part (a).

(b) $x[n] = n^2 a^n u[n]$

Starting with the definition of the z-transform,

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Differentiating both sides with respect to z yields,

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -nx[n] z^{-n-1}$$

Differentiating with respect to z again yields

$$\frac{d^2X(z)}{dz^2} = \sum_{n=-\infty}^{\infty} (-n)(-n-1)x[n]z^{-n-2}$$

Simplifying yields

$$z^2\,\frac{d^2X(z)}{dz^2} = \sum_{n=-\infty}^{\infty} n^2 x[n]z^{-n} + \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

The last term is equivalent to .

$$\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{dX(z)}{dz}$$

Therefore,

$$z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n^2 x[n]z^{-n}$$

where $X(z)$ is the z-transform of $x[n]$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n]z^{-n}$$

$$= \sum_{n=0}^{\infty}(az^{-1})^n$$
$$= \frac{1}{1-az^{-1}}$$

Evaluating the derivatives yields

$$z\frac{dX(z)}{dz} = \frac{-az^{-1}}{(1-az^{-1})^2}$$

$$z^2\frac{d^2X(z)}{dz^2} = \frac{2a^2z^{-2}}{(1-az^{-1})^3} + \frac{2az^{-1}}{(1-az^{-1})^2}$$

Adding the previous two equations and simplifying yields the final z-transform.

$$\boxed{\mathcal{Z}\{n^2a^n u[n]\} = \frac{a^2z^{-2}+az^{-1}}{(1-az^{-1})^3}}$$

Because the sequence is right sided, the ROC is

$$\boxed{|z| > |a|}$$

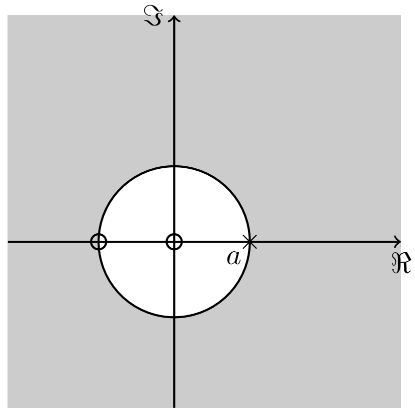

Figure 2: Pole-zero plot for part (b).

(c) $x[n] = e^{n^4} \cos\left(\frac{\pi}{12} n\right) u[n] - e^{n^4} \cos\left(\frac{\pi}{12} n\right) u[n-1]$

In this case, $x[n]$ only turns on at time $n = 0$. Therefore, the sequence can be expressed as

$$x[n] = e^{n^4} \cos\left[\frac{\pi}{12} n\right] \delta[n]$$

The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} e^{n^4} \cos\left[\frac{\pi}{12} n\right] \delta[n] z^{-n}$$

$$\boxed{X(z) = 1}$$

The ROC is all z. Also, this z transform has no poles or zeros.

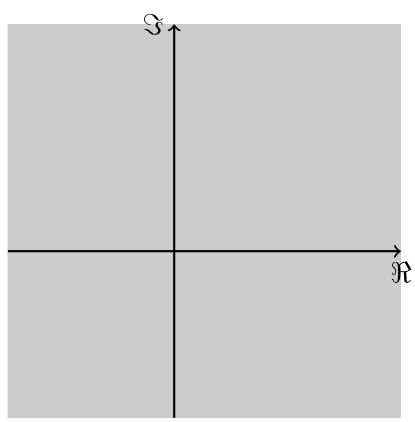

Figure 3: Pole-zero plot for part (c).

Problem 3.50

Problem Statement

Let $x[n]$ denote a causal sequence; i.e., $x[n] = 0$ for $n < 0$. Furthermore, assume that $x[0] = 0$ and that the z-transform is a rational function.

(a) Show that there are no poles or zeros of $X(z)$ at $z = \infty$ i.e., that $\lim_{z\to\infty} X(z)$ is nonzero and finite.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$).

Solution

(a) Show that there are no poles or zeros of $X(z)$ at $z = \infty$ i.e., that $\lim_{z\to\infty} X(z)$ is nonzero and finite.

For a causal sequence,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

Taking the limit as $z \to \infty$,

$$\lim_{z\to\infty} X(z) = x[0] + \cancelto{0}{x[1]z^{-1}} + \cancelto{0}{x[2]z^{-2}} + \ldots$$

Therefore,

$$\lim_{z\to\infty} X(z) = x[0]$$

Since the limit as $z \to \infty$ approaches a finite value, there cannot be any poles at infinity.

(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$).

If the z-transform is a rational function, it can be expressed as

$$X(z) = \frac{\displaystyle\sum_{k=0}^{M} b_k z^{-k}}{\displaystyle\sum_{k=0}^{N} a_k z^{-k}}$$

An equivalent expression is

$$X(z) = \frac{z^N \displaystyle\sum_{k=0}^{M} b_k z^{M-k}}{z^M \displaystyle\sum_{k=0}^{N} a_k z^{N-k}}$$

By examining the above expressions, it can be observed that the numerator is responsible for M zeros and an M-th order pole at the origin ($z = 0$). The denominator introduces N poles and an N-th order zero at the origin. Therefore, there are a total of $M + N$ poles and $M + N$ zeros. That is, the number of poles is equal to the number of zeros in the z-plane.
