AE 6345: Turbulence
Homework 4

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Problem 1

Problem Statement

(Modified 10.2)

(a) Consider the self-similar temporal mixing layer where the mean lateral velocity $\langle V \rangle = 0$ and the axial velocity $\langle U \rangle$ depends on y and t only. (Note: Such a flow is difficult to produce in practice; instead, a spatially evolving miking layer is easier to produce – FKL.) The velocity difference is U_s, so that the boundary conditions are $\langle U \rangle = 0.5U_s$ at $y = \pm\infty$. The thickness of the layer $\delta(t)$ is defined such that $\langle U \rangle = \frac{2}{5}U_s$ at $y = \delta/2$. The mixing length model is applied to this flow, with the mixing length being uniform across the flow and proportional to the thickness, namely,

$$\ell_m = \alpha\delta$$

where α is a specified constant. Starting from

$$\frac{\partial\langle U \rangle}{\partial t} = -\frac{\partial\langle uv \rangle}{\partial y}$$

show that the mixing length hypothesis implies that

$$\frac{\partial\langle U \rangle}{\partial t} = 2\alpha^2\delta^2\frac{\partial\langle U \rangle}{\partial y}\frac{\partial^2\langle U \rangle}{\partial y^2}$$

(b) Show that the above equation admits a self-similar solution of the form

$$\langle U \rangle = U_s f(\xi)$$

where $\xi = y/\delta$ and that $f(\xi)$ satisfies the ordinary differential equation

$$-S\xi f' = 2\alpha^2 f' f''$$

where $S \equiv U_s^{-1} d\delta/dt$ is the spreading rate.

(c) Justify the assumption for ℓ_m given in the first equation.

Solution

According to Prandtl's mixing-length hypothesis,

$$-\langle uv\rangle = \nu_T \frac{\partial \langle U\rangle}{\partial y}$$

and

$$\nu_T = \ell_m^2 \left|\frac{\partial \langle U\rangle}{\partial y}\right|$$

Inserting the above into the Reynolds equation,

$$\frac{\partial \langle U\rangle}{\partial t} = \frac{\partial}{\partial y}\left(\ell_m^2 \left|\frac{\partial \langle U\rangle}{\partial y}\right| \frac{\partial \langle U\rangle}{\partial y}\right) \tag{1}$$

In the mixing layer, $\partial\langle U\rangle/\partial y > 0$. So, (1) becomes

$$\frac{\partial \langle U\rangle}{\partial t} = \alpha^2\delta^2 \frac{\partial}{\partial y}\left(\left[\frac{\partial \langle U\rangle}{\partial y}\right]^2\right) \tag{2}$$

Evaluating the derivative and simplifying yields the desired result:

$$\boxed{\frac{\partial \langle U\rangle}{\partial t} = 2\alpha^2\delta^2 \frac{\partial \langle U\rangle}{\partial y}\frac{\partial^2 \langle U\rangle}{\partial y^2}} \tag{3}$$

Inserting the self similar solution, and noting that U_s is constant,

$$U_s^{-1}\frac{\partial}{\partial t}(f(\xi)) = 2\alpha^2\delta^2 \frac{\partial f(\xi)}{\partial y}\frac{\partial^2}{\partial y^2}(f(\xi)) \tag{4}$$

The differentiation will be carried out term-by-term. The first term is

$$\frac{\partial f(\xi)}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} \tag{5}$$

Since $\xi = y/\delta$ and $\delta = \delta(t)$,

$$\frac{\partial \xi}{\partial t} = -y\delta^{-2}\frac{d\delta}{dt}$$

Therefore,

$$\frac{\partial f}{\partial t} = -\xi\delta^{-1}\frac{d\delta}{dt}f'$$

and

$$U_s^{-1}\frac{\partial f(\xi)}{\partial t} = -S\xi\delta^{-1}f' \tag{6}$$

Next,

$$\frac{\partial f(\xi)}{\partial y} = \frac{df}{d\xi}\frac{\partial \xi}{\partial y} = \frac{1}{\delta}f' \tag{7}$$

The second derivative can be expanded as follows:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{df}{d\xi}\right)\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial y}\cancelto{0}{\frac{\partial^2 \xi}{\partial y^2}} = \frac{d^2 f}{d\xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 = \frac{1}{\delta^2}f'' \tag{8}$$

Inserting (6), (7), and (8) into (4),

$$-S\xi\delta^{-1}f' = 2\alpha^2\delta^2\frac{1}{\delta}f'\frac{1}{\delta^2}f'' \tag{9}$$

Simplifying,

$$\boxed{-S\xi f' = 2\alpha^2 f' f''}$$

The fact that the PDE reduces to an ODE means that the flow is self-similar.

I believe that the assumption for the length scale can be justified using the fact that the growth rate of the mixing layer is linear.

Problem 2

Problem Statement

(Modified 10.14)

(a) Consider the Spalart-Allmaras model applied to high Reynolds number homogeneous turbulence. Show that

$$\frac{\bar{D}\nu_T}{\bar{D}t} = \nabla \cdot \left(\frac{\nu_T}{\sigma_v}\nabla\nu_T\right) + S_v \tag{10}$$

reduces to

$$\frac{d\nu_T}{dt} = S_v(\nu_T, \Omega) = c_{b1}\nu_T\Omega \tag{11}$$

where c_{b1} is a constant and Ω is the mean vorticity.

The problem in the book requires an argument that the laminar viscosity ν and the distance to the nearest wall ℓ_w are not relevant quantitites. As a guide that may help you to tackle the reduction of (10) to (11), such a situation also results in isotropy. You already have experience with homogeneous turbulence, i.e., you should be able to reduce those derivatives on the left-hand side. You then need to explain (OK to explain heuristically without derivation) how that happens.

(b) Comment on the above equation for irrotational mean straining. Hint: when a flow is irrotational, is there vorticity? Think about the Kelvin-Helmholtz theorem. Think about vorticity creation in viscous flows. Then, think about the above equation. (I am deliberately making you think about this due to the large amount of erroneous thoughts regarding vorticity and vortices.)

For self-similar homogeneous turbulent shear flow (in which Ω and S are equal), from the relation defining the turbulent viscosity

$$-\langle uv\rangle = \nu_T\frac{\partial\langle U\rangle}{\partial y}$$

show that ν_T evolves with

$$c_{b1} = \left(\frac{\mathcal{P}}{\varepsilon} - 1\right)\left(\frac{\mathcal{S}k}{\varepsilon}\right)^{-1}$$

(c) Using experimental data (Table 5.4 on pg. 157), estimate c_{b1} according to the above equation.

Solution

In isotropic, homogeneous turbulence, the turbulent viscosity should be uniform. Thus, $\nabla \nu_T = 0$. Therefore,

$$\frac{d\nu_T}{dt} = S_v$$

According the the original paper by Spalart and Allmaras, $S = 0$ in isotropic turbulence so I'm not sure how the $S = c_{b1}\nu_T\Omega$ comes about. Actually, $\Omega = (2\overline{\Omega}_{ij}\overline{\Omega}_{ij})^{1/2}$ which should be zero in homogeneous, isotropic turbulence.

An irrotational flow is one in which the curl of the velocity is zero. That is

$$\epsilon_{ijk}u_{k,j} = 0$$

The rate-of-rotation tensor is

$$\Omega_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i})$$

If the fluid is *irrotational*, the rate-of-rotation tensor should be zero. Thus, $d\nu_T/dt = 0$. This agrees with what Spalart and Allmaras said in their paper.

Rearranging the expression for c_{b1},

$$c_{b1} = \frac{1}{\mathcal{S}k}(\mathcal{P} - \varepsilon)$$

Assuming $k = -\langle uv \rangle$, and differentiating both sides of the equation ($\partial\langle U\rangle/\partial y = \text{const}$),

$$\frac{dk}{dt} = \frac{d\nu_T}{dt}\frac{\partial\langle U\rangle}{\partial y}$$

Now, inserting $d\nu_T/dt = c_{b1}\nu_T\Omega$ and remembering that $\Omega = \mathcal{S}$ for this case,

$$\frac{dk}{dt} = c_{b1}\nu_T\mathcal{S}\frac{\partial\langle U\rangle}{\partial y} = c_{b1}\mathcal{S}k$$

Now, from the text,

$$\frac{dk}{dt} = \mathcal{P} - \varepsilon$$

Inserting this into the previous equation and rearranging,

$$\boxed{c_{b1} = \frac{1}{\mathcal{S}k}(\mathcal{P} - \varepsilon) = c_{b1} = \left(\frac{\mathcal{P}}{\varepsilon} - 1\right)\left(\frac{\mathcal{S}k}{\varepsilon}\right)^{-1}}$$

Using the table, three estimates for c_{b1} were determined → 0.123, 0.115, 0.093. The first two are quite close to the value quoted by Spalart and Allmaras.

Problem 3

Problem Statement

(Modified 13.2)

(a) Let $U(x)$ have the Fourier transform $\hat{U}(\kappa)$ and $\overline{U}(x)$ have the Fourier transform $\hat{\overline{U}}(x) = \hat{G}(\kappa)\hat{U}(\kappa)$. Show that the Fourier transform of the filtered residual $\overline{u}'(x)$ is

$$\hat{u}'(\kappa) \equiv \mathcal{F}\left\{u'(x)\right\} = \hat{G}(\kappa)\left[1 - \hat{G}(\kappa)\right]\hat{U}(\kappa)$$

(b) In a general three-dimensional flow, the above operations are performed on vectors, so that the residual is actually written as $\mathbf{u}'(\mathbf{x})$. Explain in your own words why the residual stress (also known as the subgrid stress) $u'v'$ is or is not the same as the Reynolds stress $\langle uv \rangle$.

(c) Consider an isotropic, homogeneous turbulence field in a three-dimensional interval $0 \leqslant \mathbf{x} \leqslant \mathbf{L}$. The text suggests a criterion for the maximum resolvable wavenumber as

$$\kappa_{\max} = \frac{\pi N_{\max}}{L}$$

based on numerical considerations, where $N_{\max}$ is the number of the uniformly spaced grid points over the dimension L. To relate this numerical relationship to physics, the text suggests that isotropic turbulence is adequately resolved with

$$\kappa_{\max}\eta \geqslant 1.5$$

where $\eta \equiv (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale, ν is the kinematic viscosity, and ε is the rate of dissipation of TKE. The dissipation decays as a power law:

$$\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0}\right)^{-(n+1)}$$

Derive an expression for the time rate of change of $\kappa_{\max}$ in terms of initial conditions and ν.

(d) Derive an expression for the grid spacing $h = L/N_{\max}$ as a function of time and the above variables. Based on your results, for decaying homogeneous, isotropic turbulence, are more grid points needed at the beginning than at the end of the computations?

Solution

$$u'(x,t) = U(x,t) - \overline{U}(x,t)$$

Taking the Fourier transform,

$$\mathcal{F}\left\{u'(x,t))\right\} = \hat{u}'(\kappa) = \hat{U}(\kappa) - \mathcal{F}\left\{\overline{U}(x,t)\right\}$$

where

$$\overline{U} = G * U$$

By the convolution theorem,

$$\mathcal{F}\{G * U\} = \hat{G}(\kappa)\hat{U}(\kappa)$$

Therefore,

$$\hat{u}'(\kappa) = \left[1 - \hat{G}(\kappa)\right]\hat{U}(\kappa) \tag{12}$$

The above is the expression for the residual. The result for the filtered residual will be derived next. The filter can be applied in the time domain via convolution or in the wavenumber domain via multiplication.

$$\overline{u}'(x,t) = G * u'$$

Taking the Fourier transform,

$$\hat{\overline{u}}'(\kappa) = \hat{G}(\kappa)\hat{u}'(\kappa)$$

Inserting (12),

$$\boxed{\hat{\overline{u}}'(\kappa) = \hat{G}(\kappa)\left[1 - \hat{G}(\kappa)\right]\hat{U}(\kappa)} \tag{13}$$

The Reynolds stress is an artifact of the averaging applied to the Navier-Stokes equations. The subgrid stress is a result of the filtering procedure applied to the Navier-Stokes. The two terms are different because the operations applied to the NS are different.

Inserting the dissipation as a function of time and rearranging yields

$$\kappa_{\max}(t) \geqslant 1.5\left(\frac{\varepsilon_0 (t/t_0)^{-(n+1)}}{\nu^3}\right)^{1/4}$$

Taking the time derivative (using Mathematica),

$$\boxed{\frac{d\kappa_{\max}}{dt} \geqslant -\frac{0.375(n+1)}{t}\left(\frac{\left(\frac{t}{t_0}\right)^{-1-n}\varepsilon_0}{\nu^3}\right)^{1/4}}$$

Inserting the expressions in the problem statement, and manipulating

$$\boxed{\frac{1}{h^2}\frac{dh}{dt} \geqslant \frac{0.375(n+1)}{\pi t}\left(\frac{\varepsilon_0\left(\frac{t}{t_0}\right)^{-(n+1)}}{\nu^3}\right)^{1/4}}$$

I believe more grid points are needed at the beginning.

Problem 4

Problem Statement

Modified (13.10)

(a) The Mansour-Wray energy spectrum can be written as

$$E(\kappa) = \frac{q^2}{2A} \frac{1}{\kappa_p^{\sigma+1} \kappa^\sigma} \exp\left[-\frac{\sigma}{2} \left(\frac{\kappa}{\kappa_p} \right)^2 \right]$$

Compare this against the expression for the Kolmogorov energy spectrum:

$$E(\kappa) = C\varepsilon^{2/3} \kappa^{-5/3}$$

The Mansour-Wray spectrum possesses certain advantages in having a number of parameters for "tweaking" the spectrum, such as q, κ_p, A and σ. For the purpose of this problem, let the Mansour-Wray spectrum be expressed as

$$E(\kappa) = \frac{B}{\kappa^\sigma} \exp(-\kappa^2)$$

Consider the use of a sharp spectral filter whose spectral characteristics are defined by

$$\hat{G}(\kappa) = \begin{cases} 1 & |\kappa| < \kappa_c \\ 0 & |\kappa| \geqslant \kappa_c \end{cases}$$

Apply this filter to the Mansour-Wray spectrum and obtain an expression for the energy in the residual motion (i.e., the energy in the subgrid scal), $\langle k_r \rangle$.

(b) Let $\sigma = 5/3$. Suppose that it is desired to capture 80% of the energy via LES; in other words,

$$\frac{\langle k_r \rangle}{k} = 0.2$$

Introducing the turbulent lengthscale L and the Kolmogorov timescale τ_η, such that $k = (\nu L/\tau_\eta)^{2/3}$, simplify the above expression.

(c) Further simplify the result by letting $\exp(-\kappa_c^2) = 1$ (i.e., assume that κ_c is infinitely large) to obtain an expression for $\kappa_c L$.

(d) Defining the filter width $\Delta = \pi/\kappa_c$ and writing $\ell_{EI} = aL$ where a is a constant, obtain an expression for the filter width in terms of ℓ_{EI}.

Solution

The Mansour-Wray spectrum seems much more complicated than the Kolmogorov energy spectrum. The filter can be applied via multiplication in the wavenumber domain.

$$\overline{E}(\kappa) = \hat{G}(\kappa)^2 \frac{B}{\kappa^\sigma} \exp(-\kappa^2)$$

The energy in the subgrid scale should be the energy that was filtered out in the above equation. That is,

$$\langle k_r \rangle = \int_0^\infty (E(\kappa) - \overline{E}(\kappa))\, d\kappa = \int_0^\infty \left(1 - [\hat{G}(\kappa)]^2\right) \frac{B}{\kappa^\sigma} \exp(-\kappa^2)\, d\kappa$$

Introducing the turbulent lengthscale and Kolmogorov timescale along with the given expression for k,

$$\langle k_r \rangle = \frac{1}{5} k = \frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \int_0^\infty \left(1 - [\hat{G}(\kappa)]^2\right) \frac{B}{\kappa^{5/3}} \exp(-\kappa^2)\, d\kappa$$

Further simplification can be carried out by assuming that $\kappa_c \to \infty$ (i.e., $\exp(-\kappa_c^2) \to 1$). This will be applied after integrating. The transfer function for the sharp spectral filter is

$$\hat{G}(\kappa) = H(\kappa_c - |\kappa|)$$

where $H(\kappa)$ is a Heaviside step function. Since $H(\kappa)^2 = H(\kappa)$,

$$\frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \int_0^\infty (1 - H(\kappa_c - \kappa))\, B\kappa^{-5/3} \exp(-\kappa^2)\, d\kappa$$

The absolute value of κ was dropped since the bounds of integration don't include any negative values. Furthermore, $1 - H(\kappa) = H(-\kappa)$.

$$\frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \int_0^\infty H(\kappa - \kappa_c) B\kappa^{-5/3} \exp(-\kappa^2)\, d\kappa$$

Using Mathematica to evaluate the integral yields

$$\frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \frac{3}{2} \left(\frac{e^{-{\kappa_c}^2}}{{\kappa_c}^{2/3}} - \Gamma\left(\frac{2}{3}, {\kappa_c}^2 \right) \right) = \frac{3}{2} \left(\frac{1}{{\kappa_c}^{2/3}} - \Gamma\left(\frac{2}{3}, {\kappa_c}^2 \right) \right)$$

where Γ stands for the incomplete gamma function:

$$\Gamma(a, x) \equiv \int_x^\infty t^{a-1} e^{-t}\, dt$$

So,

$$\frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \frac{3}{2} \left(\frac{1}{{\kappa_c}^{2/3}} - \int_{\kappa_c^2}^\infty t^{-1/3} e^{-t}\, dt \right)$$

Again, assuming that $\kappa_c^2 \to \infty$,

$$\frac{1}{5} \left(\frac{\nu L}{\tau_\eta} \right)^{2/3} = \frac{3}{2} \left(\frac{1}{{\kappa_c}^{2/3}} \right)$$

Rearranging,

$$\boxed{\kappa_c L = \frac{20.5396 B^{3/2} \tau_\eta}{\nu}}$$

Inserting $\kappa_c = \pi/\Delta$, $L = \ell_{EI}/a$ and rearranging,

$$\boxed{\Delta = \frac{\ell_{EI} \pi \nu}{20.5396\, a\, B^{3/2} \tau_\eta}}$$