AE 5342 - Project 2

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Contents

1 Problem Statement

A piston moves from rest towards the right with constant acceleration $\dot{u}$ in a quiescent gas having initially uniform temperature and pressure. As the gas is compressed, the right running characteristics converge (see Figure 7.18 of the textbook). Assuming inviscid dynamics, a shock forms at the earliest intersection of characteristics of the same family (C^+ in this case). Your task is to determine the location of the shock formation as a function of the isentropic index.

Include in a typed report:

- The governing equations.
- A suitable dimensionless form of the same equations.
- The solution procedure.
- The non-dimensional shock formation location $\overline{x}$ and time $\overline{t}$ as a function of γ only.
- A graph (not a sketch) of the dimensionless velocity field in a neighborhood of the shock location for $\gamma = 7/5$.

- Assuming that the gas is air in standard conditions $T = 298$ K and $p = 1$ atm, the numerical value of the acceleration $\dot{u}$ such that a shock forms within 1 meter of the initial resting location of the piston.

2 Solution

2.1 Governing Equations

2.1.1 Fundamental Governing Equations

Assumptions:

1. Isentropic flow
2. Inviscid dynamics
3. Quasi-one dimensional flow

The continuity equation, Eq. (6.22) from Anderson, is

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{V}) = 0$$

Remembering from thermodynamics that any state variable can be expressed as a function of two other state variables,

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds \tag{1}$$

Remembering the first assumption of isentropic flow, (1) can be written in terms of the material derivative as

$$\frac{D\rho}{Dt} = \frac{1}{a^2}\frac{Dp}{Dt} \tag{2}$$

For one-dimensional flow, (2) becomes

$$\frac{1}{a^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0 \tag{3}$$

Considering the momentum equation, Eq. (6.29) in Anderson,

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla p$$

Applying (6.22) to a one-dimensional flow,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{4}$$

Adding and subtracting (3) and (4) yields

$$\left[\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right] = 0$$

$$\text{and} \tag{5}$$

$$\left[\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x}\right] - \frac{1}{\rho a}\left[\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right] = 0$$

Solution of the equations in (5) yields $u(x,t)$ and $p(x,t)$. These equations are solved using the method of characteristics as outlined in Chapter 7 of Anderson's book.

2.1.2 Method of Characteristics

As the piston accelerates, C^+ characteristics propagate from the piston face. The slopes of these C^+ characteristic lines are given by

$$\boxed{\frac{dx}{dt} = u + a} \tag{6}$$

The Riemann invariants are

$$\boxed{\begin{aligned} J^+ &= u + \frac{2a}{\gamma - 1} \\ J^- &= u - \frac{2a}{\gamma - 1} \end{aligned}} \tag{7}$$

The position of the piston face as a function of the acceleration ($\dot{u}$) and time (t) is given by

$$\boxed{x_p = \frac{1}{2}\,\dot{u}_p\, {t_p}^2} \tag{8}$$

The velocity is given by

$$\boxed{u = \dot{u}_p\, t_p} \tag{9}$$

Using the fact that J^- is constant everywhere and the fact that $u_\infty = 0$,

$$J^- = u - \frac{2a}{\gamma - 1} = \cancelto{0}{u_\infty} - \frac{2a_\infty}{\gamma - 1}$$

$$\therefore$$

$$a = \frac{\gamma - 1}{2} u + a_\infty \tag{10}$$

Also, solving (9) for t_p,

$$t_p = \frac{u}{\dot{u}_p} \tag{11}$$

Inserting (11) into (8),

$$x_p = \frac{1}{2}\frac{u^2}{\dot{u}_p} \tag{12}$$

Now, using the fact that the C^+ characteristics are straight lines, (6) becomes

$$\frac{x - x_p}{t - t_p} = (u + a) \tag{13}$$

Inserting (10), (11) and (12) into (13) and rearranging,

$$x - \frac{1}{2}\frac{u^2}{\dot{u}_p} = \left(u + \frac{\gamma - 1}{2}\, u + a_\infty\right)\left(t - \frac{u}{\dot{u}_p}\right) \tag{14}$$

Now, the dimensionless parameters will be determined, and then (14) will be nondimensionalized.

2.2 Dimensionless form of Governing Equations

2.2.1 Buckingham Pi Analysis

The parameters we are concerned with are x, u, $\dot{u}$, a_∞, and t. Looking at the dimensions of the variables:

x	u	$\dot{u}$	a_∞	t
L	LT^{-1}	LT^{-2}	LT^{-1}	T

The number of fundamental dimensions is two (i.e., $j = 2$). Therefore, using the Buckingham Pi Theorem,

$$k = n - j = 5 - 2 = 3$$

where k is the number of dimensionless groups we can expect from the analysis and n is the total number of variables. So, we should expect three dimensionless groups. Choosing the repeating variables to be a_∞ and $\dot{u}$, the first Π group is

$$\Pi_1 = (a_\infty)^a(\dot{u})^b x = (\mathrm{LT}^{-1})^a(\mathrm{LT}^{-2})^b(\mathrm{L}) = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0$$

$$\begin{aligned} \text{Length:} &\quad a + b + 1 = 0 \\ \text{Time:} &\quad -a - 2b = 0 \end{aligned} \;\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Solution of the linear system of equations yields $a = -2$ and $b = 1$. Therefore, the first dimensionless group is given by

$$\boxed{\overline{x} = \Pi_1 = \frac{\dot{u}\, x}{{a_\infty}^2}} \tag{15}$$

This Π group represents the dimensionless position denoted by $\overline{x}$.

Similarly,

$$\Pi_2 = (a_\infty)^a(\dot{u})^b\, t = (\mathrm{LT}^{-1})^a(\mathrm{LT}^{-2})^b(\mathrm{T}) = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0$$

$$\begin{aligned} \text{Length:} &\quad a + b = 0 \\ \text{Time:} &\quad -a - 2b + 1 = 0 \end{aligned} \;\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$a = -1 \quad \text{and} \quad b = 1$$

$$\therefore$$

$$\boxed{\bar{t} = \Pi_2 = \frac{t}{a_\infty/\dot{u}}} \tag{16}$$

and

$$\Pi_3 = (a_\infty)^a (\dot{u})^b\, u = (\mathrm{LT}^{-1})^a (\mathrm{LT}^{-2})^b (\mathrm{LT}^{-1}) = \mathrm{M}^0\mathrm{L}^0\mathrm{T}^0$$

$$\begin{array}{rr} \text{Length:} & a + b + 1 = 0 \\ \text{Time:} & -a - 2b - 1 = 0 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$a = -1 \quad \text{and} \quad b = 0$$

$$\therefore$$

$$\boxed{\overline{u} = \Pi_3 = \frac{u}{a_\infty}} \tag{17}$$

2.2.2 Nondimensionalizing the Governing Equations

Expanding (14),

$$x = a_\infty t + \frac{1}{2}ut - \frac{a_\infty\, u}{\dot{u}_p} - \frac{\gamma u^2}{2\dot{u}_p} + \frac{\gamma u t}{2}$$

Multiplying both sides by $\dot{u}_p$,

$$\dot{u}_p x = a_\infty \dot{u}_p t + \frac{1}{2}u\dot{u}_p t - a_\infty u - \frac{\gamma}{2}u^2 + \frac{\gamma u \dot{u}_p t}{2}$$

Dividing both sides by a_∞^2,

$$\frac{\dot{u}_p x}{a_\infty^2} = \frac{\dot{u}_p t}{a_\infty} + \frac{1}{2}\frac{u\dot{u}_p t}{a_\infty^2} - \frac{u}{a_\infty} - \frac{\gamma}{2}\frac{u^2}{a_\infty^2} + \frac{\gamma}{2}\frac{u\dot{u}_p t}{a_\infty^2} \tag{18}$$

Each term in (18) is dimensionless. Using the parameters defined in Eqs. (15) - (17), Eq. (18) becomes

$$\overline{x} = \bar{t} + \frac{1}{2}\overline{u}\bar{t} - \overline{u} - \frac{\gamma}{2}\overline{u}^2 + \frac{\gamma}{2}\overline{u}\bar{t}$$

Rearranging,

$$\boxed{\frac{\gamma}{2}\,\overline{u}^2 + \left[1 - \left(\frac{\gamma+1}{2}\right)\bar{t}\,\right]\overline{u} + (\overline{x} - \bar{t}) = 0} \tag{19}$$

Eq. (19) will be solved using the quadratic equation next.

2.3 Procedure

Outline:

1. Determine governing equations.
2. Use Method of Characteristics to change the form of the governing partial differential equations to ordinary differential equations.
3. Determine kinematic relationships.
4. Algebraically manipulate expressions.
5. Determine dimensionless parameters.
6. Nondimensionalize the resulting expression from step 4 by further algebraic manipulation.
7. Solve for the mass motion $\overline{u}(\overline{x}, \overline{t})$ using the quadratic formula.
8. Plot the gradients of $\overline{u}(\overline{x}, \overline{t})$ on a surface plot of the function (see Figure 1).
9. Solve for the $\overline{x}$ and $\overline{t}$ coordinates of the area where the gradients become very large.
10. Create animation of change in velocity profile with time.

Solving (19) yields two solutions:

$$\overline{u}(\overline{x}, \overline{t}) = \begin{pmatrix} \dfrac{\overline{t} + 2\sqrt{\dfrac{\gamma^2\,\overline{t}^2}{4} + \dfrac{\gamma\,\overline{t}^2}{2} + \gamma\,\overline{t} - 2\,\gamma\,\overline{x} + \dfrac{\overline{t}^2}{4} - \overline{t} + 1} + \gamma\,\overline{t} - 2}{2\,\gamma} \\ \dfrac{\overline{t} - 2\sqrt{\dfrac{\gamma^2\,\overline{t}^2}{4} + \dfrac{\gamma\,\overline{t}^2}{2} + \gamma\,\overline{t} - 2\,\gamma\,\overline{x} + \dfrac{\overline{t}^2}{4} - \overline{t} + 1} + \gamma\,\overline{t} - 2}{2\,\gamma} \end{pmatrix}$$

Evaluating both solutions at $\overline{u}(0, 0)$, it is evident that the first solution is valid because $u(0, 0)$ must be zero. Therefore,

$$\overline{u}(\overline{x}, \overline{t}) = \frac{\overline{t} + 2\sqrt{\dfrac{\gamma^2\,\overline{t}^2}{4} + \dfrac{\gamma\,\overline{t}^2}{2} + \gamma\,\overline{t} - 2\,\gamma\,\overline{x} + \dfrac{\overline{t}^2}{4} - \overline{t} + 1} + \gamma\,\overline{t} - 2}{2\,\gamma} \tag{20}$$

2.4 Shock Formation Location

The shock forms where the gradients of pressure, density, and velocity become very large. Figure 1 shows a plot of $\overline{u}(\overline{x}, \overline{t})$ with the colormap set as $\nabla\overline{u}$. The green dashed line is the first characteristic. The solid green line is the path of the piston. Behind the solid green line (i.e., in the space behind the piston), the function given in Eq. (20) is no longer valid. We are concerned with the region between the green lines. As $\overline{x}$ and $\overline{t}$ increase, more characteristics propagate from the face of the piston.

Solving (20) for $\overline{x}$,

$$\overline{x}(\overline{u}, \overline{t}) = \overline{t} - \overline{u} + \frac{1}{2}\overline{u}\,\overline{t} - \frac{\gamma}{2}\overline{u}^2 + \frac{\gamma}{2}\overline{u}\,\overline{t}$$

The shock forms where $\partial\overline{u}/\partial\overline{x} \to \infty$. Equivalently, the shock forms where $\partial\overline{x}/\partial\overline{u} \to 0$. Finding the partial of $\overline{x}$ with respect to $\overline{u}$,

$$\frac{\partial\overline{x}}{\partial\overline{u}} = \left(\frac{\gamma+1}{2}\right)\overline{t} - \gamma\,\overline{u} - 1$$

Assuming that the shock forms at the first intersection of the initial characteristic and another characteristic that propagated from the moving piston face, zero induced mass motion along the initial characteristic is imposed by setting $\overline{u} = 0$. Therefore,

$$\left(\frac{\gamma+1}{2}\right)\overline{t}_{\text{shock}} - 1 = 0$$

$$\therefore$$

$$\boxed{\overline{t}_{\text{shock}} = \frac{2}{\gamma+1} \approx 0.833} \tag{21}$$

The equation of the first characteristic in dimensionless form is $\overline{x} = \overline{t}$. Therefore the $\overline{x}$-coordinate of the shock is given by

$$\boxed{\overline{x}_{\text{shock}} = \frac{2}{\gamma+1} \approx 0.833} \tag{22}$$

The results in (21) and (22) agree well with Figure 1.

2.5 Dimensionless Velocity Field

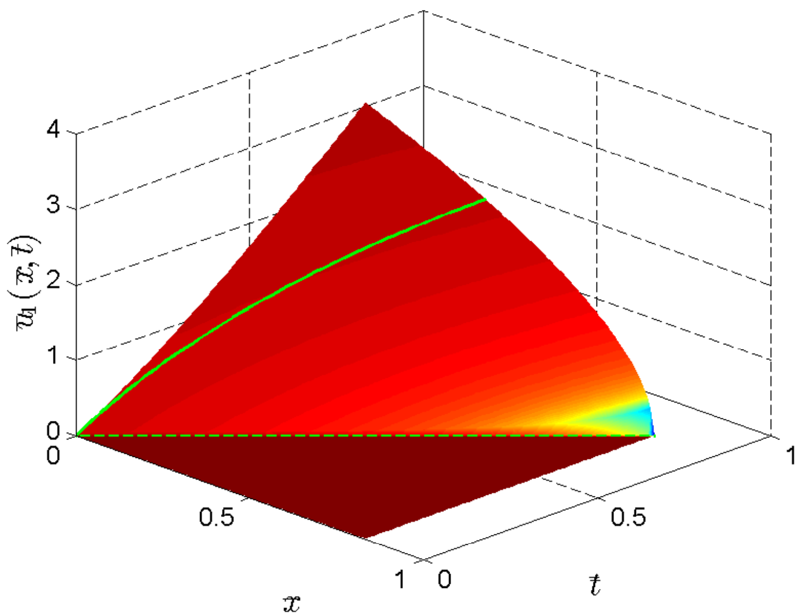

Figure 1. Eq. (20) with colormap as gradient of $\overline{u}(\overline{x}, \overline{t})$.

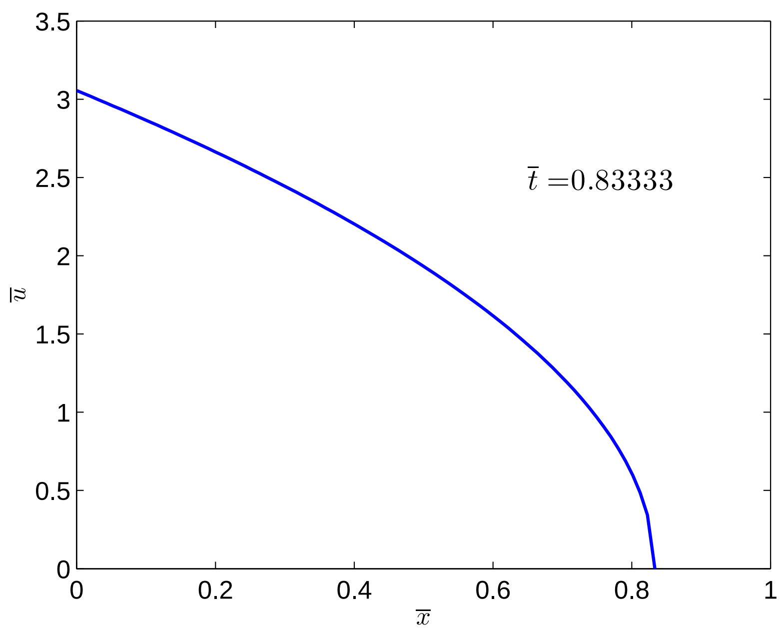

Figure 2. Dimensionless velocity distribution at $t = t_{\text{shock}}$.

Out of curiosity, a quick animation of the variation of the velocity profile with time was created using MATLAB. The video can be seen by typing the following link into your browser (the tilde is causing problems with the link):

`http://omega.uta.edu/˜jrg2179/GD_Proj_2.html`

2.6 Bonus

Eqs. (15) and (22) will be used to determine the value of the acceleration needed to create a shock within 1 meter of the initial resting location of the piston. Remembering the definition of the speed of sound,

$$a_\infty^2 = \gamma R T$$

Equating (15) and (22),

$$\overline{x}_{\text{shock}} = \frac{\dot{u} x_{\text{shock}}}{a_\infty^2} = \frac{2}{\gamma + 1}$$

$$\therefore$$

$$\boxed{\dot{u} = \frac{\gamma R T}{x_{\text{shock}}} \left(\frac{2}{\gamma + 1} \right) = 99780 \text{ m/s}^2} \tag{23}$$

A MATLAB code

Filename: `Project_2.m`

```
%% AE 5342 Project 2
% James Grisham
% April 9, 2013

clear,clc,close all

%-----------------------------------------------------------------------
%% Inputs
%-----------------------------------------------------------------------

% Setting defaults
set(0,'defaulttextinterpreter','LaTeX')
set(0,'defaultaxesfontname','Helvetica')
set(0,'DefaultAxesFontSize',12)

% Setting up path for saving images
ImgPath = ['C:\Users\James\Desktop\School\Courses\',...
    'UTA\AE 5342 - Gas Dynamics\Images\'];

% Plot decision
plotdec = 1;

% Isentropic index
g = 7/5;

%-----------------------------------------------------------------------
%% Calculations
%-----------------------------------------------------------------------
mStop = 2/(g + 1);
mStart = 0;

syms a a_inf u x t x_p ud_p t_p gamma

% From J-
a = (gamma-1)/2*u + a_inf;

% From kinematics
t_p = u/ud_p;
x_p = 1/2*u.^2/ud_p;

% Substituting and manipulating
disp('-----------------------------------------------------------------')
disp('Dimensional form:')
disp('-----------------------------------------------------------------')
xl = expand((u + a).*(t - t_p) + x_p);
disp('x = ')
pretty(xl)
xl = subs(xl,a_inf,1);
xl = subs(xl,ud_p,1);

fprintf('\n\n')
disp('-----------------------------------------------------------------')
disp('Dimensionless form:')
```

```
disp('-----------------------------------------------------------------')
fprintf('\n\n')
disp('x(u,t,gamma) = ')
pretty(xl)

% Setting up quadratic
myexp = xl - x;
fprintf('\n')
disp('Manipulating:')
pretty(-myexp)
disp(' = 0')
% latex(-myexp)
fprintf('\n')
disp('-----------------------------------------------------------------')
disp('Solution to quadratic equation:')
disp('-----------------------------------------------------------------')
fprintf('\n\n')
disp('u(x,t,gamma) = ')
u = solve(myexp == 0,u);
pretty(u)

% Separating solutions
u1(x,t) = u(1);
u2(x,t) = u(2);

clear u
syms u
fprintf('\n\n')
disp('-----------------------------------------------------------------')
disp('Solving for x(u,t,gamma):')
disp('-----------------------------------------------------------------')
fprintf('\n')
disp('x(u,t,gamma) = ')
xexp = solve(u1 == u,x);
pretty(xexp)

fprintf('\n\n')
disp('-----------------------------------------------------------------')
disp('Solving for the location of shock formation:')
disp('-----------------------------------------------------------------')
fprintf('\n')

% Finding the partial derivative of x WRT u
disp('x_u = ')
pretty(diff(xexp,u))
t_shock = solve(diff(xexp,u)==0,t);

% Because the induced mass motion along the first characteristic must be
% zero
t_shock = subs(t_shock,u,0);
disp('t_shock = ')
pretty(t_shock)
fprintf('\n')
x_shock = subs(t_shock,gamma,g);
t_shock = subs(t_shock,gamma,g);

fprintf('\n')
disp('-----------------------------------------------------------------')
disp('Bonus:')
```

```
disp('-------------------------------------------------------------------')

x_s = 1;                                         % m
R = 287;                                         % J/(kg*K)
T = 298;                                         % K
udot = g*R*T/x_s*(2/(g + 1));
fprintf(['\nFor the shock to form within 1 m of the initial',...
    ' resting \nlocation of the piston, udot = %5.0f m/s^2.\n\n'],udot)



% Plotting

if plotdec == 1

    figure
    set(gcf,'Renderer','zbuffer')
    view(45,30)
    hold on

    % Setting up
    Nelements = 80;
    gamma = 7/5;
    tvec = linspace(0,t_shock,Nelements);
    xvec = linspace(0,x_shock,Nelements);
    [x,t] = meshgrid(xvec,tvec);
    u = t + 2*sqrt(gamma^2*t.^2./4 + gamma*t.^2./2 + gamma*t ...
        - 2*gamma*x + t.^2/4 - t + 1) + gamma*t - 2;

    % Path of piston
    t_p = tvec;
    x_p = 1/2*t_p.^2;
    u_p = t_p + 2*sqrt(gamma^2*t_p.^2./4 + gamma*t_p.^2./2 + gamma*t_p ...
        - 2*gamma*x_p + t_p.^2/4 - t_p + 1) + gamma*t_p - 2;

    % First characteristic
    t_1 = tvec;
    x_1 = xvec;
    u_1 = zeros(1,numel(x_1));

    % Enforcing zero mass motion in the uniform region
    u(u<0) = 0;

    grid
    surf(x,t,u,gradient(u),'EdgeColor','none')
    shading interp
    colormap(jet)
    set(gca,'GridLineStyle','--')

    xlabel('$\overline{x}$')
    ylabel('$\overline{t}$')
    zlabel('$\overline{u}_1 \, (\,\overline{x},\overline{t})$')

    h_p = plot3(x_p,t_p,u_p,'-g','LineWidth',1.5);
    h_1 = plot3(x_1,t_1,u_1,'--g','LineWidth',1.5);

    % Saving plot
    set(gcf,'PaperPositionMode','auto')
    print(gcf, '-depsc', [ImgPath,'Proj2_surf_1.eps'])
```

```

% Animation

    t = tvec;
    x = xvec;
    figure
    set(gca,'XLim',[0 1])

    % Setting x and y locations for t display
    xloc = 0.65;
    yloc = 2.5;

    for n = 1:numel(tvec)
        u = t(n) + 2*sqrt(gamma^2*t(n)^2./4 + gamma*t(n)^2./2 + gamma*t(n) ...
            - 2*gamma*x + t(n)^2/4 - t(n) + 1) + gamma*t(n) - 2;
        u(u<0)=0;
        if n == 1
            h = plot(x,u,'-b','LineWidth',1.5);
            xlabel('$\overline{x}$')
            ylabel('$\overline{u}$')
            set(gca,'YLim',[0 3.5])
            th = text(xloc,yloc,['$\overline{t} = $',num2str(t(n))],...
                'FontSize',14);
            Film(n) = getframe(gcf); %#ok<*SAGROW>
        else
            set(h,'YDataSource','u')
            refreshdata
            pause(.01)
            set(th,'String',['$\overline{t} = $',num2str(t(n))])
            Film(n) = getframe(gcf);
        end
    end

    % Saving movie
    movie2avi(Film,[ImgPath,'Project_2.avi'],'Compression','Cinepak',...
        'Quality',100)

    % Saving plot
    set(gcf,'PaperPositionMode','auto')
    print(gcf, '-depsc', [ImgPath,'Proj2_ux.eps'])

end
```