AE 5311 - Take Home Final Exam

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5/9/2013

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Problem 1

Problem Statement

For an under-damped SDOF system (m, k, c), use the Laplace transform method to derive solution to

(**a**). Initial conditions (u_0, v_0)

(**b**). Step input (P_0, zero initial conditions)

(**c**). Sine input ($P_s \sin \Omega\, t$, zero initial conditions)

(**d**). Cosine input ($P_c \cos \Omega\, t$, zero initial conditions)

Express your solution in terms of the given inputs and the usual system parameters (ω_n, ζ, ω_d, etc.).

Solution

Part A

For the given system, the EOM is

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{1}$$

Now, taking the Laplace transform of (1),

$$\begin{aligned} \mathcal{L}[m\ddot{u}] &= m\left(s^2U(s) - su(0) - \dot{u}(0)\right) \\ \mathcal{L}[c\dot{u}] &= c\left(sU(s) - u(0)\right) \\ \mathcal{L}[ku] &= kU(s) \end{aligned} \tag{2}$$

Inserting (2) into (1) and noting that $u(0) = u_0$ and $\dot{u}(0) = v_0$,

$$ms^2U(s) - msu_0 - mv_0 + csU(s) - cu_0 + kU(s) = 0$$

Rearranging,

$$\left(ms^2 + cs + k\right) U(s) = msu_0 + mv_0 + cu_0$$

Dividing by both sides by m,

$$\left(s^2 + \frac{c}{m}\, s + {\omega_n}^2\right) U(s) = su_0 + v_0 + \frac{c}{m}\, u_0$$

$\because c/m = 2\omega_n\zeta$,

$$\left(s^2 + 2\omega_n\zeta + {\omega_n}^2\right) U(s) = su_0 + v_0 + 2\omega_n\zeta u_0$$

Solving for $U(s)$,

$$U(s) = \frac{u_0(s + 2\omega_n\zeta)}{s^2 + 2\omega_n\zeta + {\omega_n}^2} + \frac{v_0}{s^2 + 2\omega_n\zeta + {\omega_n}^2}$$

Taking the inverse Laplace transform to determine $u(t)$,

$$u(t) = \mathcal{L}^{-1}[U(s)] = \frac{v_0}{\omega_d}\, e^{-\zeta\omega_n t} \sin\omega_d t + u_0\, e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{(1-\zeta^2)^{1/2}} \sin\omega_d t\right) \tag{3}$$

where the inverse Laplace transforms were found using the table in Appendix C. Simplifying (3),

$$\boxed{u(t) = e^{-\zeta\omega_n t}\left(u_0 \cos\omega_d t + \frac{v_0 + \zeta\omega_n u_0}{\omega_d} \sin\omega_d t\right)} \tag{4}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

This result agrees perfectly with Eq. (3.30) from the textbook.

Part B

Find the response to a step input, P_0.

For the given system, the EOM is

$$m\ddot{u} + c\dot{u} + ku = P_0 \tag{5}$$

Now, taking the Laplace transform of (5),

$$\begin{aligned}
\mathcal{L}[m\ddot{u}] &= m\left(s^2U(s) - s\cancelto{0}{u(0)} - \cancelto{0}{\dot{u}(0)}\right) \\
\mathcal{L}[c\dot{u}] &= c\left(sU(s) - \cancelto{0}{u(0)}\right) \\
\mathcal{L}[ku] &= kU(s) \\
\mathcal{L}[P_0] &= \frac{P_0}{s}
\end{aligned} \tag{6}$$

Inserting (6) into (5) and simplifying,

$$U(s) = \frac{P_0}{k}\frac{{\omega_n}^2}{s(s^2 + 2\omega_n\zeta + {\omega_n}^2)} \tag{7}$$

Finding the poles of (7),

$$\lambda_1 = 0, \quad \lambda_2 = \omega_n\left(-\zeta + i\sqrt{1-\zeta^2}\right), \quad \text{and} \quad \lambda_2 = \omega_n\left(-\zeta - i\sqrt{1-\zeta^2}\right)$$

Using Mathematica to perform partial fractions and simplify the resulting expression,

$$u(t) = \frac{P_0 e^{\zeta(-t)\omega_n}\left(\sqrt{1-\zeta^2}\left(\omega_d \cos\left(\sqrt{1-\zeta^2}t\omega_n\right) + \zeta\omega_n \sin(t\omega_d)\right) + \left(\zeta^2-1\right)\omega_n e^{\zeta t\omega_n}\right)}{\left(\zeta^2-1\right)k\omega_n}$$

Simplifying further using $\omega_d = \omega_n\sqrt{1-\zeta^2}$, we arrive at the same equation that is in the book.

$$\boxed{u(t) = \frac{P_0}{k}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right]} \tag{8}$$

The equation for a step response of an SDOF system from the textbook (Eq. (5.5)) is

$$u(t) = \frac{P_0}{k}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right]$$

To make sure the equations from Mathematica and the textbook were the same, they were plotted against each other.

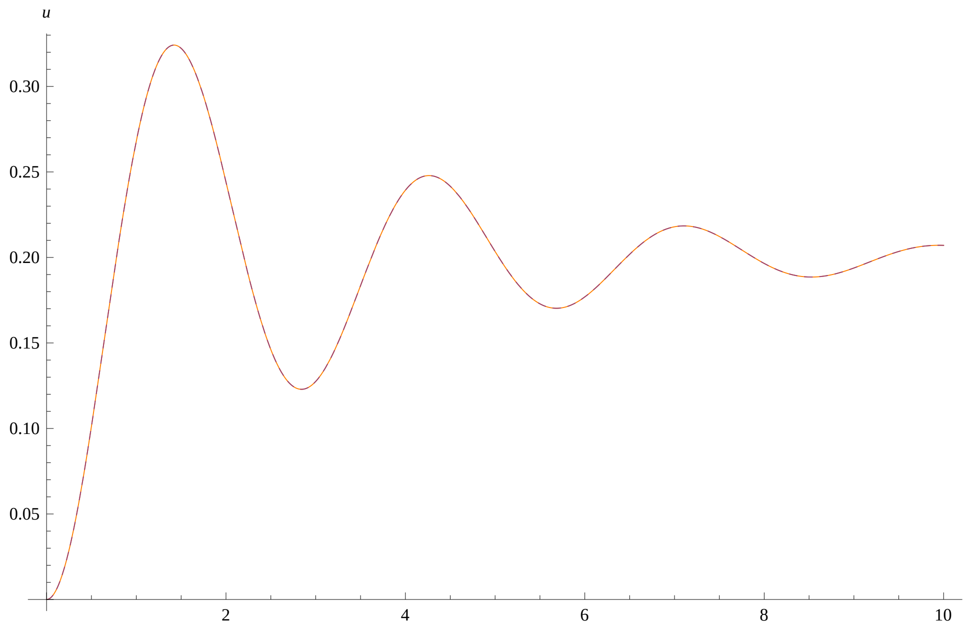


Figure 1. Comparison between book equation and my solution.

Part C

Find the response to a Sine input.

The EOM for the system is

$$m\ddot{u} + c\dot{u} + ku = P_s \sin \Omega_s t$$

Taking the Laplace transform,

$$m\left(s^2 U(s) + su(0) + \dot{u}(0)\right) + c\left(sU(s) + u(0)\right) + kU(s) = \frac{P_s \Omega_s}{s^2 + \Omega_s^2}$$

Applying zero initial conditions,

$$ms^2 U(s) + csU(s) + kU(s) = \frac{P_s \Omega_s}{s^2 + \Omega_s^2}$$

Solving for $U(s)$,

$$U(s) = \frac{P_s \Omega_s}{\left(s^2 + \Omega_s^2\right)\left(cs + k + ms^2\right)}$$

Finding the poles,

$$s = \{i\Omega_s, \quad -i\Omega_s, \quad -\zeta\omega_n - i\omega_d, \quad -\zeta\omega_n + i\omega_d\}$$

Using a partial fraction decomposition,

$$U(s) = \frac{A}{s - \zeta\omega_n - i\omega_d} + \frac{B}{s - \zeta\omega_n + i\omega_d} + \frac{C}{s + i\Omega_s} + \frac{D}{s - i\Omega_s} \tag{9}$$

Where

$$\begin{aligned} A &= (s - \zeta\omega_n - i\omega_d)\, U(s)\big|_{s=\zeta\omega_n + i\omega_d} \\ B &= (s - \zeta\omega_n + i\omega_d)\, U(s)\big|_{s=\zeta\omega_n - i\omega_d} \\ C &= (s + i\Omega_s)\, U(s)\big|_{s=-i\Omega_s} \\ D &= (s - i\Omega_s)\, U(s)\big|_{s=i\Omega_s} \end{aligned} \tag{10}$$

So that

$$\begin{aligned} A &= \frac{\omega_n^2 \Omega_s P_s / k}{(2i\omega_d)[(\zeta\omega_n + i\omega_d)^2 + \Omega_s^2]} \\ B &= \frac{\omega_n^2 \Omega_s P_s / k}{(-2i\omega_d)[(\zeta\omega_n - i\omega_d)^2 + \Omega_s^2]} \\ C &= \frac{\omega_n^2 \Omega_s P_s / k}{(-2i\Omega_s)[(-i\Omega_s - \zeta\omega_n)^2 + \omega_d^2]} \\ D &= \frac{\omega_n^2 \Omega_s P_s / k}{(2i\Omega_s)[(i\Omega_s - \zeta\omega_n)^2 + \omega_d^2]} \end{aligned}$$

Taking the inverse Laplace transform of (9),

$$u(t) = Ae^{(\zeta\omega_n + i\omega_d)t} + Be^{(\zeta\omega_n - i\omega_d)t} + Ce^{-i\Omega_s t} + De^{i\Omega_s t}$$

This equation does not match the book equation perfectly, but it should give the same result.

Part D

The EOM for the system is

$$m\ddot{u} + c\dot{u} + ku = P_c \cos \Omega_c t$$

Taking the Laplace transform,

$$m\left(s^2 U(s) + s u(0) + \dot{u}(0)\right) + c\left(sU(s) + u(0)\right) + kU(s) = \frac{sP_c}{\Omega_c^2 + s^2}$$

Applying zero initial conditions,

$$ms^2 U(s) + csU(s) + kU(s) = \frac{sP_c}{\Omega_c^2 + s^2}$$

Solving for $U(s)$,

$$U(s) = \frac{sP_c}{\left(\Omega_c^2 + s^2\right)\left(cs + k + ms^2\right)}$$

Inserting relations for ω_n and ζ,

$$U(s) = \frac{sP_c\,\omega_n^2}{k\left(\Omega_c^2 + s^2\right)\left(2\zeta s\omega_n + \omega_n^2 + s^2\right)}$$

Expanding,

$$\begin{aligned} U(s) \quad &= \quad \frac{-2\zeta P_c\omega_n^5 + sP_c\Omega_c^2\omega_n^2 - sP_c\omega_n^4}{k\left(2\zeta s\omega_n + \omega_n^2 + s^2\right)\left(4\zeta^2\Omega_c^2\omega_n^2 - 2\Omega_c^2\omega_n^2 + \Omega_c^4 + \omega_n^4\right)} \\ &\quad + \frac{2\zeta P_c\Omega_c^2\omega_n^3 - sP_c\Omega_c^2\omega_n^2 + sP_c\omega_n^4}{k\left(\Omega_c^2 + s^2\right)\left(4\zeta^2\Omega_c^2\omega_n^2 - 2\Omega_c^2\omega_n^2 + \Omega_c^4 + \omega_n^4\right)} \end{aligned}$$

Taking the inverse Laplace transform, and simplifying

$$
\begin{aligned}
u(t) \quad = \quad & -\frac{P_c\omega_n^2\left(\omega_n^2-\Omega_c^2\right)e^{2it\omega_d-\left(\zeta+i\sqrt{1-\zeta^2}\right)t\omega_n}}{2k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& +\frac{\zeta\sqrt{1-\zeta^2}P_c\omega_n^4\sin\left(t\omega_d\right)e^{it\omega_d-\left(\zeta+i\sqrt{1-\zeta^2}\right)t\omega_n}}{\left(\zeta^2-1\right)k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& +\frac{\zeta\sqrt{1-\zeta^2}P_c\Omega_c^2\omega_n^2\sin\left(t\omega_d\right)e^{it\omega_d-\left(\zeta+i\sqrt{1-\zeta^2}\right)t\omega_n}}{\left(\zeta^2-1\right)k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& -\frac{P_c\omega_n^4e^{\left(\zeta+i\sqrt{1-\zeta^2}\right)(-t)\omega_n}}{2k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& +\frac{P_c\Omega_c^2\omega_n^2e^{\left(\zeta+i\sqrt{1-\zeta^2}\right)(-t)\omega_n}}{2k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& +\frac{2\zeta P_c\Omega_c\omega_n^3\sin\left(t\Omega_c\right)}{k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)} \\
& +\frac{P_c\omega_n^4\cos\left(t\Omega_c\right)}{k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)}-\frac{P_c\Omega_c^2\omega_n^2\cos\left(t\Omega_c\right)}{k\left(4\zeta^2\Omega_c^2\omega_n^2-2\Omega_c^2\omega_n^2+\Omega_c^4+\omega_n^4\right)}
\end{aligned}
$$

The symbolic computations were accomplished using Mathematica. This form is not fully simplified, but the result should give the same answer as the previously known solutions.

Problem 2

Problem Statement

For an undamped SDOF system, use Laplace transform method to derive solution due to (**a**). Sine resonant response ($P_s \sin \omega_n t$) with zero initial conditions.

(**b**). Cosine resonant response ($P_c \cos \omega_n t$) with zero initial conditions.

Solution

Part A

The EOM for the given system is

$$m\ddot{u} + ku = P_s \sin \omega_n t$$

Taking the Laplace transform of both sides,

$$m\left(s^2U(s) + \cancelto{0}{su(0)} + \cancelto{0}{\dot{u}(0)}\right) + kU(s) = \frac{P_s\,\omega_n}{s^2 + \omega_n^2}$$

Solving for $U(s)$,

$$\left(ms^2 + k\right)U(s) = \frac{P_s\,\omega_n}{s^2 + \omega_n^2}$$

$$U(s) = \frac{P_s\,\omega_n}{(ms^2 + k)(s^2 + \omega_n^2)}$$

Manipulating,

$$U(s) = \frac{P_s\,\omega_n}{m(s^2 + \omega_n^2)(s^2 + \omega_n^2)}$$

$$U(s) = \frac{P_s}{m}\,\frac{\omega_n}{(s^2 + \omega_n^2)^2}$$

Dividing the numerator and denominator by k,

$$U(s) = \frac{P_s\,\omega_n^3}{k}\,\frac{1}{(s^2 + \omega_n^2)^2}$$

Taking the inverse Laplace transform,

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\{U(s)\} \\ &= \frac{P_s\,\omega_n^3}{k}\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + \omega_n^2)^2}\right\} \\ &= \frac{P_s\,\omega_n^3}{k}\left(\frac{1}{2\omega_n^3}\left[\sin \omega_n t - \omega_n t \cos \omega_n t\right]\right) \end{aligned}$$

$$\boxed{u(t) = \frac{P_s}{2k}\left(\sin\omega_n t - \omega_n t \cos\omega_n t\right)} \tag{11}$$

This result matches previously derived results using the limiting process.

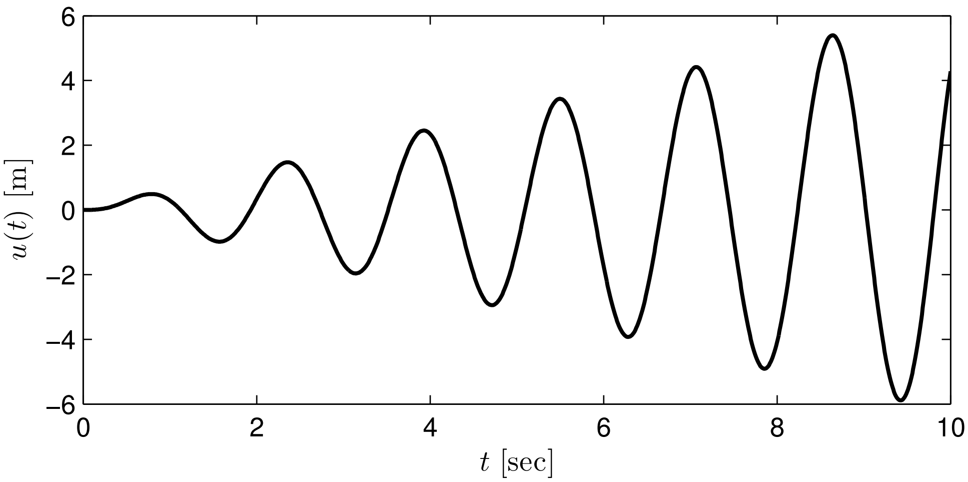


Figure 2. Plot of Eq. (11).

Part B

The EOM for the given system is

$$m\ddot{u} + ku = P_c \cos\omega_n t$$

Taking the Laplace transform of both sides,

$$m\left(s^2U(s) + \cancelto{0}{su(0)} + \cancelto{0}{\dot{u}(0)}\right) + kU(s) = \frac{P_c\, s}{s^2+\omega_n^2}$$

Dividing by m,

$$s^2U(s) + \omega_n^2 U(s) = \frac{P_c}{m}\,\frac{s}{s^2+\omega_n^2}$$

Solving for $U(s)$,

$$U(s) = \frac{P_c}{m}\,\frac{s}{(s^2+\omega_n^2)^2}$$

Simplifying further by dividing the numerator and denominator by k,

$$U(s) = \frac{\omega_n^2\, P_c}{k}\,\frac{s}{(s^2+\omega_n^2)^2}$$

Taking the inverse Laplace transform,

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\left\{U(s)\right\} \\ &= \frac{\omega_n^2\,P_c}{k}\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega_n^2)^2}\right\} \\ &= \frac{\omega_n^2 P_c}{k}\left(\frac{t\sin\omega_n t}{2\omega_n}\right) \\ &\therefore \end{aligned}$$

$$\boxed{u(t) = \frac{P_c}{2k}\left(\omega_n t\right)\sin\omega_n t} \tag{12}$$

Again, the result in (12) matches known results perfectly (see Eq. (4.15) from Craig).

Problem 3

Problem Statement

Find the response of a SDOF system with $m = 2$ kg, $k = 32$ N/m subject to a saw tooth periodic input shown below. Also plot the solution, use enough terms to convince yourself the solution has converged. Find the response at $t = 4$ seconds.

(*Hint:* Express the periodic input as a Fourier series and then treat each term in the series as a forcing function and use the formulas derive in Problem 1 to find solution to each term. The total solution is the sum of the solution due to all terms.)

Solution

First, the triangular wave must be represented as a Fourier series. The definition of a Fourier series is as follows. The Fourier series corresponding to a function $f(t)$ defined in the interval $c \le t \le c + 2L$ where c and $L > 0$ are constants is defined as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_c^{c+2L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_c^{c+2L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt$$

Since the function is odd (i.e., $f(-t) = -f(t)$),

$$a_n = 0 \quad \text{and} \quad a_0 = 0$$

Examining the Triangular wave, it is symmetric about the vertical and horizontal axes. Because of this symmetry, it is only necessary to integrate over a quarter of the period.

$$f(t) = \frac{4P_0}{T_0} t \ \text{ for } \ 0 \le t \le \frac{T_0}{4}$$

$$\therefore$$

$$b_n = \int_0^{\frac{T_0}{4}} \frac{4P_0 t}{T_0} \sin\left(\frac{2\pi n t}{T_0}\right) dt + \int_{\frac{T_0}{4}}^{\frac{3T_0}{4}} \left(2P_0 - \frac{4P_0 t}{T_0}\right) \sin\left(\frac{2\pi n t}{T_0}\right) dt + \int_{\frac{3T_0}{4}}^{T_0} \left(\frac{4P_0 t}{T_0} - 4P_0\right) \sin\left(\frac{2\pi n t}{T_0}\right) dt$$

Evaluating the integral,

$$b_n = \frac{P_0 T_0 \left(2\sin\left(\frac{\pi n}{2}\right) - 2\sin\left(\frac{3\pi n}{2}\right) + \sin(2\pi n)\right)}{\pi^2 n^2}$$

for $n = 1, 3, 5, \ldots$, $\sin(2\pi n) = 0$. Also, $\sin(\pi n/2) = -\sin(3\pi n/2)$ =1, −1, 1 for $n = 1, 3, 5$. Therefore, b_n becomes

$$b_n = \begin{cases} \dfrac{4T_0P_0}{n^2\pi^2} & \text{if } n = 1,\ 5,\ 9,\ \ldots \\[2ex] \dfrac{-4T_0P_0}{n^2\pi^2} & \text{if } n = 3,\ 7,\ 11,\ \ldots \\[2ex] 0 & \text{otherwise} \end{cases} \tag{13}$$

So, the triangular function expressed as an infinite sum of sinusoids is given by,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi t}{T_0}\right) \tag{14}$$

where b_n is given by Eq. (13).

To make sure the derived Fourier series was correct, it was plotted with MATLAB. The results are shown in Figure 3. The series was implemented in MATLAB as follows:

```
for n = 1:nTerms

    % Sequence for odd numbers
    k = 2*n-1;
    fprintf('k = %d\n',k)

    % Logic structure to change sign of b_n term
    if k == 4*counter - 3
        b(n) = 4*P_0*T_0/(pi^2*k^2);
        counter = counter+1;
    else
        b(n) = -4*P_0*T_0/(pi^2*k^2);
    end

    % Summing terms
    sum = sum + b(n)*sin(2*k*pi*t/T_0);

    % Forcing frequency
    Omega(n) = 2*k*pi/T_0;

end
```

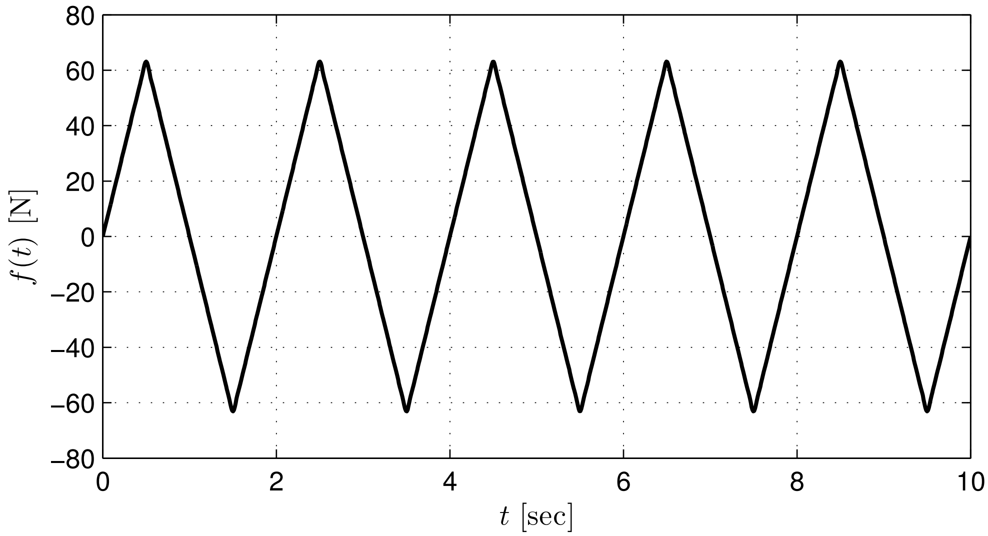


Figure 3. Triangular wave from 5-term expansion given by Eq. (14).

Now, writing out the first five terms of Eq. (14), the forcing function is given by

$$f(t) \approx \frac{4P_0T_0}{\pi^2}\sin\left(\frac{2\pi t}{T_0}\right) - \frac{4P_0T_0}{9\pi^2}\sin\left(\frac{6\pi t}{T_0}\right) + \frac{4P_0T_0}{25\pi^2}\sin\left(\frac{10\pi t}{T_0}\right) - \frac{4P_0T_0}{49\pi^2}\sin\left(\frac{14\pi t}{T_0}\right) + \frac{4P_0T_0}{81\pi^2}\sin\left(\frac{18\pi t}{T_0}\right) \tag{15}$$

Assigning each amplitude to A_i for $i = 1$ to 5, and assigning Ω_i to the interior terms of the sin functions, excluding time,

$$f(t) \approx A_1 \sin\Omega_1 t + A_2 \sin\Omega_2 t + A_3 \sin\Omega_3 t + A_4 \sin\Omega_4 t + A_5 \sin\Omega_5 t$$

For the given system, the EOM is given by

$$m\ddot{u} + ku = \sum_{i=1}^{N} A_i \sin\Omega_i t$$

where $N = 5$, for this case. Using the previously derived solution of an

SDOF system to a sine input and setting $c = 0$,

$$\boxed{u(t) \approx \sum_{i=1}^{N} \frac{A_i}{k} \frac{\omega_n}{\omega_n^2 - \Omega_i^2} \left(\omega_n \sin \Omega_i t - \Omega_i \sin \omega_n t\right)} \tag{16}$$

MATLAB implementation:

```
for n = 1:nTerms

    % Equation for response
    u = u + A(n)/k*omega_n/(omega_n^2 - Omega(n)^2)*...
        (omega_n*sin(Omega(n)*t) - Omega(n)*sin(omega_n*t));

end
```

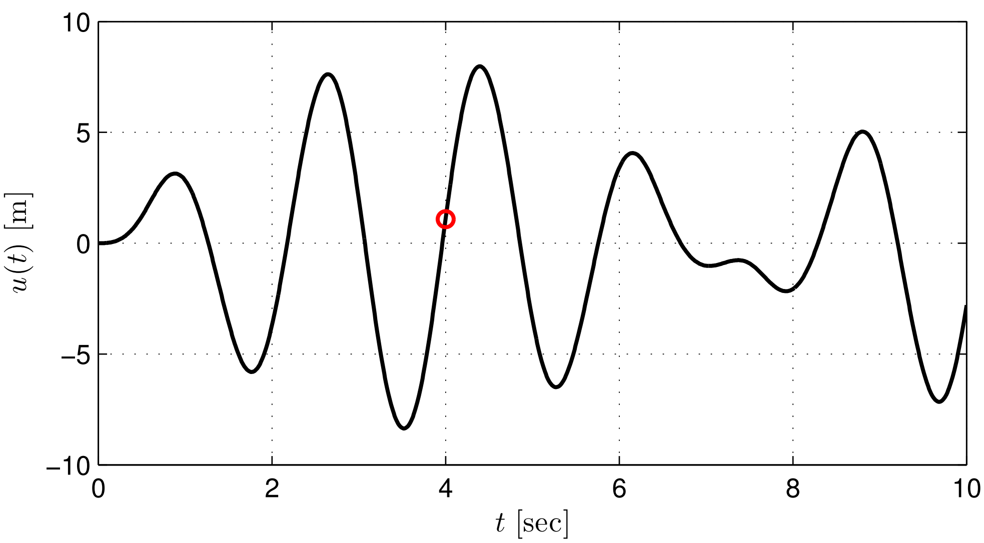


Figure 4. Response of undamped SDOF system to triangular wave loading.

Using the MATLAB function, interp1, to determine the displacement of the system at $t = 4$ by linear interpolation,

$$\boxed{u(4) \approx 1.08 \text{ m}}$$

Problem 4

Problem Statement

Find the response of the system of Problem 3 to a saw-tooth pulse shown below.

(**a**). Find the solution by Duhamel integral or any other analytical method.

(**b**). Plot the response for $t \leq T_0$. Also plot the response for this time period using the solution of Problem 3.

(**c**). Find the response at $t = 10$ seconds.

(**d**). Validate the analytical solution by ode45.

Solution

Part A

From Eq. (5.36) from the text, for zero initial conditions,

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau)\, \sin(\omega_n(t-\tau))\, d\tau$$

For the first interval, $0 \leq t \leq T_1$ and

$$p(\tau) = \frac{P_0}{T_1}\, \tau$$

The Duhamel integral becomes,

$$u_1(t) = \frac{P_0}{m\omega_n T_1} \int_0^t \tau \sin(\omega_n(t-\tau))\, d\tau$$

Evaluating the integral,

$$\boxed{u_1(t) = \frac{P_0}{m\omega_n^3 T_1}\, (\omega_n t - \sin\omega_n t)}$$

For the second interval, $T_1 \leq t \leq 3T_1$,

$$p(\tau) = 2P_0 - \frac{P_0 \tau}{T_1}$$

So, the Duhamel integral becomes,

$$u_2(t) = \int_0^{T_1} \frac{P_0\tau \sin(\omega_n(t-\tau))}{mT_1\omega_n}\, d\tau + \int_{T_1}^{t} \frac{\left(2P_0 - \frac{P_0\tau}{T_1}\right)\sin(\omega_n(t-\tau))}{m\omega_n}\, d\tau$$

Evaluating the integral,

$$\boxed{u(t) = -\frac{P_0\left(\omega_n\left(t-2T_1\right) - 2\sin\left(\omega_n\left(t-T_1\right)\right) + \sin\left(t\omega_n\right)\right)}{mT_1\omega_n^3}}$$

For the third interval, $3T_1 \le t \le 4T_1$,

$$p(\tau) = \frac{P_0\tau}{T_1} - 4P_0$$

So, the Duhamel integral becomes,

$$\begin{aligned} u_3(t) &= \int_0^{T_1} \frac{P_0\tau \sin(\omega_n(t-\tau))}{mT_1\omega_n}\, d\tau + \int_{T_1}^{3T_1} \frac{\left(2P_0 - \frac{P_0\tau}{T_1}\right)\sin(\omega_n(t-\tau))}{m\omega_n}\, d\tau \\ &\quad + \int_{3T_1}^{t} \frac{\left(\frac{P_0\tau}{T_1} - 4P_0\right)\sin(\omega_n(t-\tau))}{m\omega_n}\, d\tau \end{aligned}$$

Evaluating the integral,

$$\boxed{u(t) = -\frac{P_0\left(-\omega_n\left(t-4T_1\right) - 2\sin\left(\omega_n\left(t-T_1\right)\right) + 2\sin\left(\omega_n\left(t-3T_1\right)\right) + \sin\left(t\omega_n\right)\right)}{mT_1\omega_n^3}}$$

For the last interval,

$$\begin{aligned} u_4(t) &= \int_0^{T_1} \frac{P_0\tau \sin(\omega_n(t-\tau))}{mT_1\omega_n}\, d\tau + \int_{T_1}^{3T_1} \frac{\left(2P_0 - \frac{P_0\tau}{T_1}\right)\sin(\omega_n(t-\tau))}{m\omega_n}\, d\tau \\ &\quad + \int_{3T_1}^{4T_1} \frac{\left(\frac{P_0\tau}{T_1} - 4P_0\right)\sin(\omega_n(t-\tau))}{m\omega_n}\, d\tau \end{aligned}$$

Evaluating the integral,

$$\boxed{u_4(t) = \frac{P_0\left(2\sin\left(\omega_n\left(t-T_1\right)\right) - 2\sin\left(\omega_n\left(t-3T_1\right)\right) + \sin\left(\omega_n\left(t-4T_1\right)\right) - \sin\left(t\omega_n\right)\right)}{mT_1\omega_n^3}}$$

Part B

The responses as computed by Fourier series method, the Duhamel integral method and the numerical method (ode45) are depicted below.

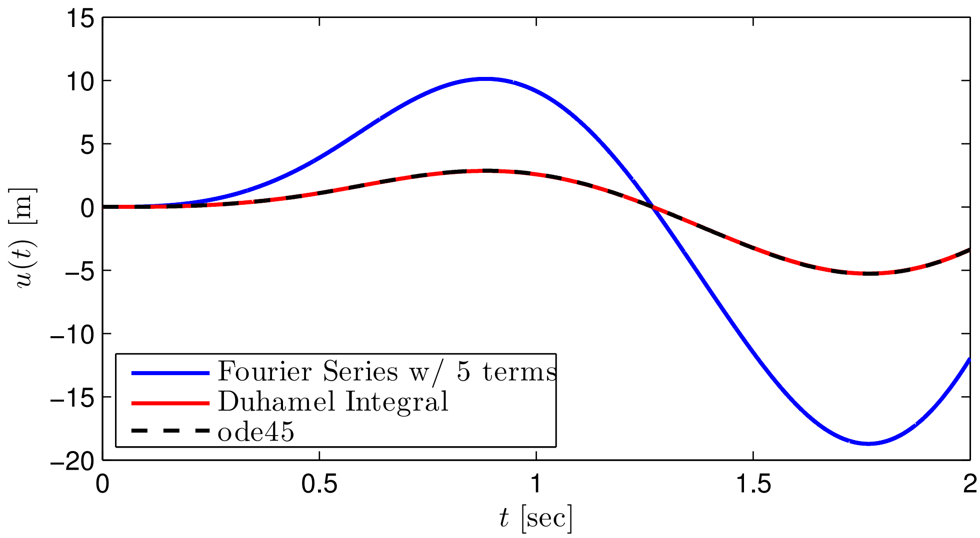


Figure 5. Comparison between solution methods with Fourier series expansion of 5 terms.

Because the 5-term Fourier series didn't agree well with the numerical and analytical solutions, the MATLAB program was generalized to allow for an arbitrary number of terms. After some trial and error, a 16-term expansion was chosen because of sufficient convergence.

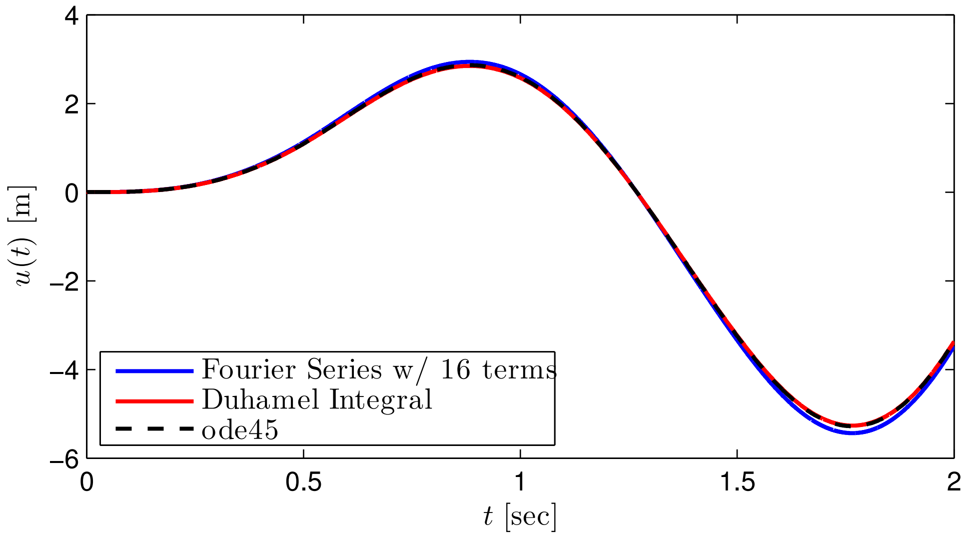


Figure 6. Comparison between solution methods with Fourier series expansion of 16 terms.

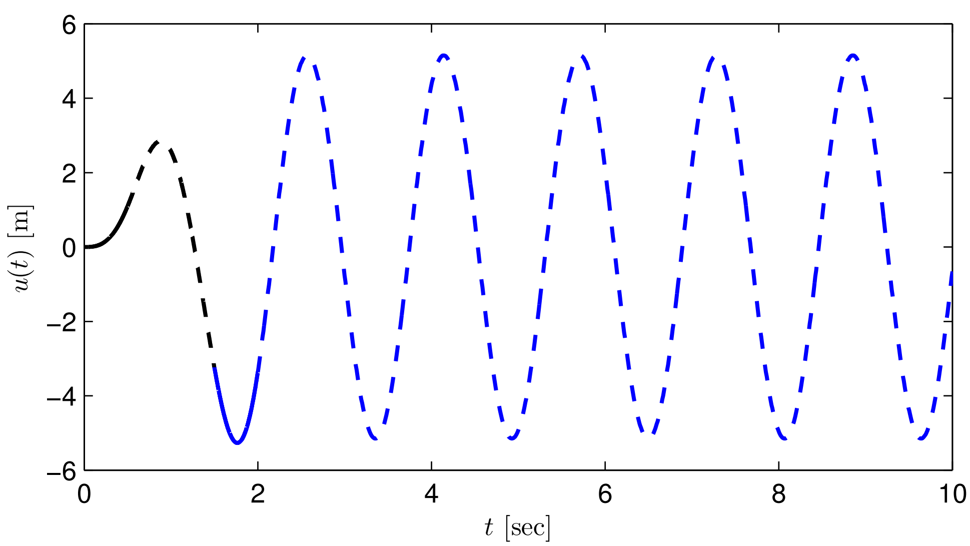


Figure 7. Response of SDOF to triangular wave.

Part C

$$\boxed{u(10) = -0.6591 \text{ m}}$$

Part D

The previous solutions were validated by using MATLAB's built-in ODE solver, ode45. The EOM for the given system is

$$m\ddot{u} + ku = T(t) \tag{17}$$

where $T(t)$ represents the triangular wave form. For this to be implemented in MATLAB, this second order ODE must be reduced to a system of two first order odes. This process is accomplished below:

$$x_1 = u$$

$$x_1' = \dot{u}$$

$$x_2 = x_1' = \dot{u}$$

$$x_2' = \ddot{u}$$

Now, rewriting (17),

$$x_2' + \omega_n^2 x_1 = \frac{1}{m} T(t)$$

Solving for x_2',

$$\boxed{x_2' = \frac{1}{m} T(t) - \omega_n x_1} \tag{18}$$

Eq. (18) is implemented in a MATLAB function as follows:

```
function dx = func(t,x)
    dx = zeros(2,1);
    dx(1) = x(2);
    dx(2) = triangleWave(t)/m - omega_n^2*x(1);
end
```

To provide ode45 with the correct triangular wave input, a function was written that created the correct loading for only the first period (see below code).

```matlab
function Twave = triangleWave(t)

    % Preallocating Twave
    Twave = zeros(1,numel(t));

    % Setting up loop
    for n = 1:numel(t)

        if t(n) <= T_0/4
            Twave(n) = 4*P_0/T_0*t(n);
        elseif t(n) > T_0/4 && t(n) <= 3*T_0/4
            Twave(n) = -4*P_0/T_0*t(n) + 2*P_0;
        elseif t(n) > 3*T_0/4 && t(n) <= 5*T_0/4
            Twave(n) = 4*P_0/T_0*t(n) - 4*P_0;
        else
            break
        end

    end
end
```

Problem 5

Problem Statement

For a 3-DOF system with the following data:

$$\mathbf{K} = \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 400 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The system is proportionally damped with damping factors 0.1, 0.2, and 0.3, respectively for modes 1, 2, and 3.

(**a**). Find the analytical solution of the total response to the following initial conditions:

$$\mathbf{u}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

(**b**). Find the steady-state solution of the system due to a unit sinusoidal load applied at DOF 2.

(**c**). Plot $U_1(\Omega)$ using CO-QUAD and Nyquist plots.

(**d**). Plot $U_2(\Omega)$ using Bode plots.

(**e**). Find amplitude of the responses at $\Omega = (\omega_1 + \omega_2)/2$.

(**f**). Check part (**d**) by direct solution.

Solution

Part A

The EOM of the system is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t) \tag{19}$$

Given the mass and stiffness matrices, the eigenvectors and eigenvalues are determined by solving the generalized eigenvalue problem.

$$\mathbf{K}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Lambda}$$

After finding the eigenvalues, the natural frequency of the r-th mode is given by

$$\omega_r = \sqrt{\lambda_r}$$

Next, the generalized mass and stiffness matrices are determined.

$$\mathcal{M} = \boldsymbol{\Phi}^\top \mathbf{M}\boldsymbol{\Phi}$$

$$\mathcal{K} = \boldsymbol{\Phi}^\top \mathbf{K}\boldsymbol{\Phi}$$

The generalized damping matrix is given by

$$\mathcal{C} = \boldsymbol{\Phi}^\top \mathbf{C}\boldsymbol{\Phi} = \text{diag}(2\zeta_r\omega_r\mathcal{M}_r)$$

Now, the equation of motion can be written in modal coordinates as

$$\mathcal{M}\ddot{\mathbf{q}} + \mathcal{C}\dot{\mathbf{q}} + \mathcal{K}\mathbf{q} = \boldsymbol{\Phi}^\top \mathbf{p}(t)$$

Since this is a 3-DOF system, we now have 3 uncoupled ODEs in modal coordinates. The initial conditions must also be transformed for modal coordinates as follows.

$$\left.\begin{aligned} q_r(0) &= \frac{1}{\mathcal{M}_r}\boldsymbol{\phi}_r^\top \mathbf{M}\mathbf{u}(0) \\ \dot{q}_r(0) &= \frac{1}{\mathcal{M}_r}\boldsymbol{\phi}_r^\top \mathbf{M}\dot{\mathbf{u}}(0) \end{aligned}\right\} \qquad r = 1,\ 2,\ \ldots,\ N$$

```
First uncoupled differential equation:
2.942 q'' + 3.324 q' + 93.92 q = 0.000

Second uncoupled differential equation:
1.529 q'' + 7.280 q' + 216.59 q = 0.000

Third uncoupled differential equation:
1.322 q'' + 13.193 q' + 365.59 q = 0.000
```

Solution of these ordinary differential equations yields

```
u_1(t) =
0.02786*exp(-0.565*t)*sin(5.622*t) - ...
    0.008462*exp(-2.38*t)*sin(11.66*t) - ...
    0.003655*exp(-4.988*t)*sin(15.86*t)

u_2(t) =
0.03311*exp(-4.988*t)*sin(15.86*t) + 0.03101*exp(-2.38*t)*sin(11.66*t) ...
    + 0.02014*exp(-0.565*t)*sin(5.622*t)

u_3(t) =
0.02657*exp(-2.38*t)*sin(11.66*t) - 0.02165*exp(-4.988*t)*sin(15.86*t) ...
    + 0.005992*exp(-0.565*t)*sin(5.622*t)
```

Next, the responses of each DOF are plotted (see Figure 8).

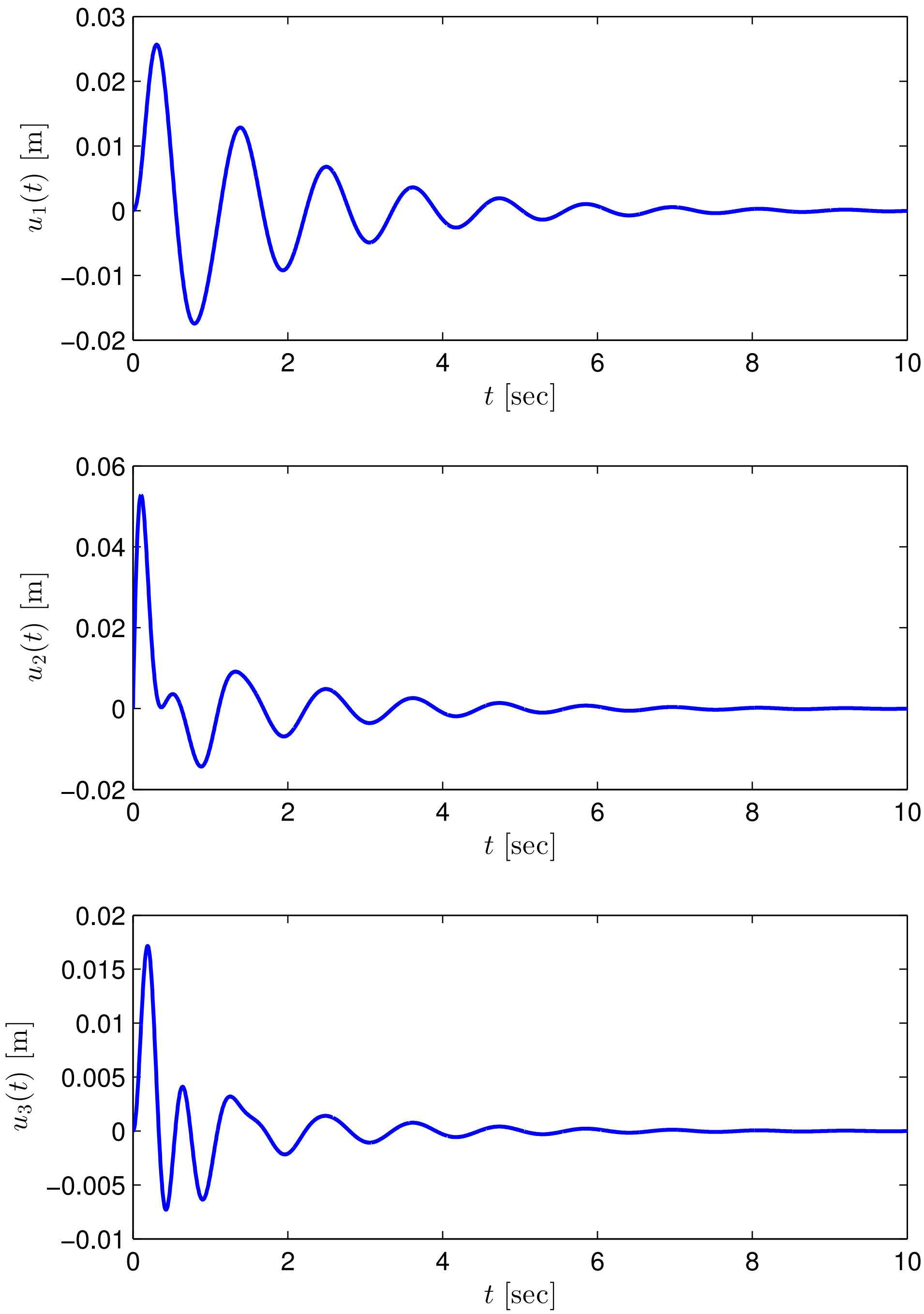


Figure 8. Response of proportionally damped system to initial conditions.

Part B

For this problem $\mathbf{u}(0) = \{0\}$, $\dot{\mathbf{u}}(0) = \{0\}$, and

$$\mathbf{p}(t) = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \cos \Omega t$$

The complex frequency response function $\overline{H}_{ij}(\Omega)$ is given in the textbook by Eq. (11.31).

$$\overline{H}_{ij}(\Omega) = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{jr}}{K_r} \frac{1}{(1 - r_r^2) + i(2\zeta_r r_r)}$$

where H_{ij} represents the response at coordinate u_i due to harmonic excitation at p_j.

$$\tan \alpha_r = \frac{2\zeta_r r_r}{1 - r_r^2} \quad \text{where} \quad r_r = \frac{\Omega}{\omega_r}$$

To find the steady state solution of the system at DOF 2, i is set to 2 and because there is only harmonic excitation at DOF 2, $j = 2$. Therefore,

$$\boxed{\overline{H}_{22} = \sum_{r=1}^{3} \frac{\phi_{2r}\phi_{2r}}{K_r} \frac{1}{\left(1 - \frac{\Omega^2}{\omega_r^2}\right) + i\left(2\zeta_r \frac{\Omega}{\omega_r}\right)}} \tag{20}$$

Using MATLAB to perform the calculations:

```
N_1/D_1(W) =

0.003547/(- 0.03132*W^2 + W*(0.0354*i + 0.0) + 1.0)

N_2/D_2(W) =

0.002553/(- 0.007061*W^2 + W*(0.03361*i + 0.0) + 1.0)

N_3/D_3(W) =

0.0019/(- 0.003617*W^2 + W*(0.03609*i + 0.0) + 1.0)
```

Part C

To plot $U_1(\Omega)$, the time domain data for the first degree of freedom from Part A was transformed into the frequency domain using the Fast Fourier Transform or FFT. This was implemented in MATLAB as follows:

```
% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4bFRF.eps'])

%% Part C - using fft to find U_1(Omega)

% Determining time step
Deltat = t_span(2) - t_span(1);

% Sampling frequency
f_s = 1/Deltat;

% Frequency resolution
Deltaf = 1/t_final;

% Number of samples
u_1 = keepu_1;
```

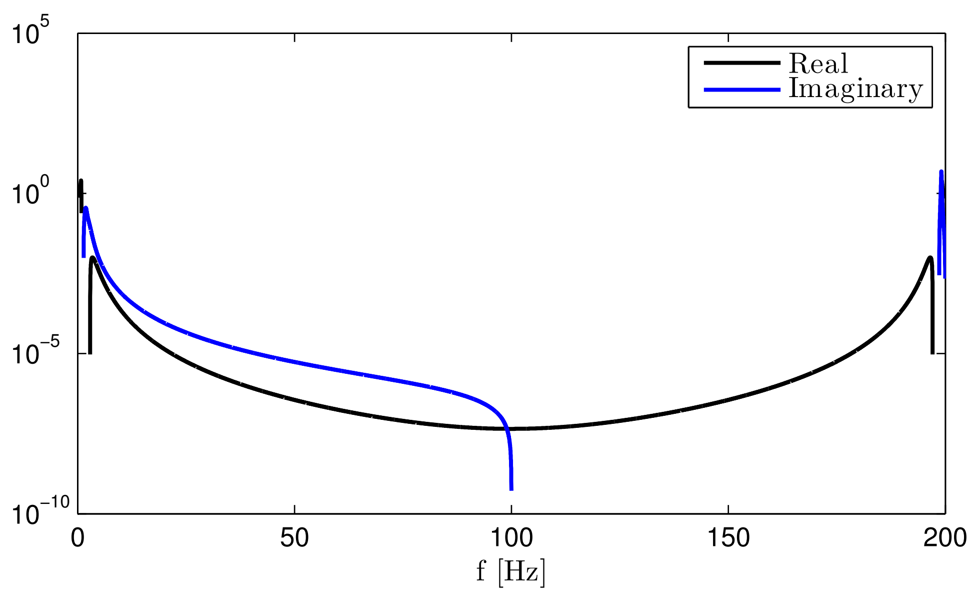


Figure 9. CO-QUAD plot.

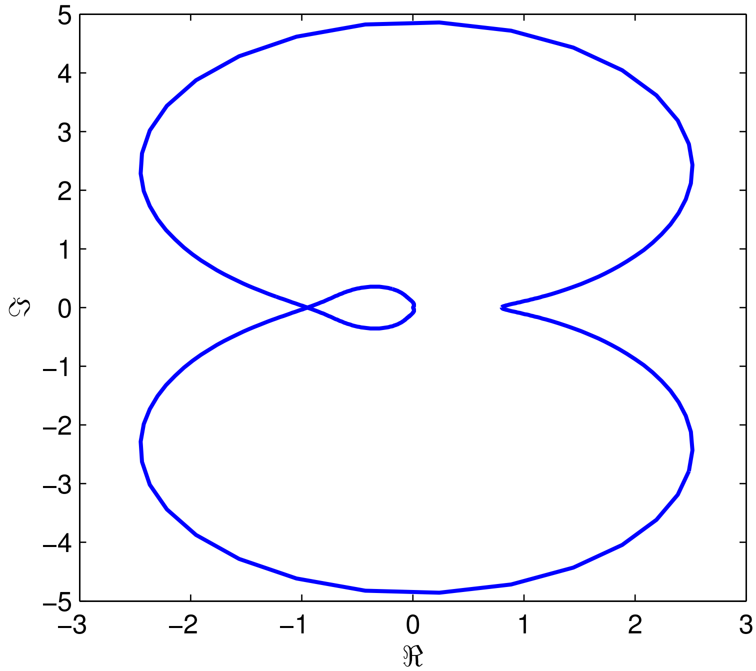


Figure 10. Nyquist plot.

Part D

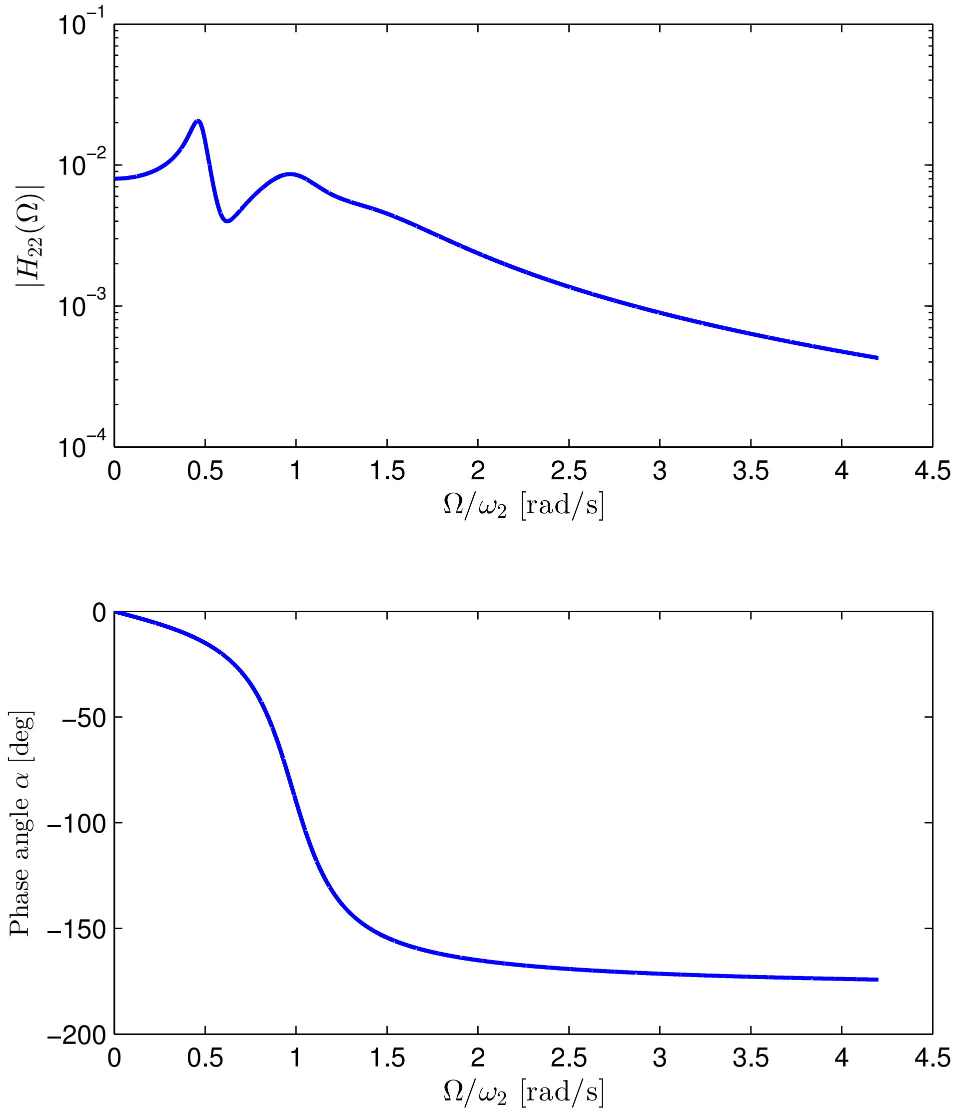


Figure 11. Bode plot of Eq. (20).

Part E

To determine the amplitudes of the responses at $\Omega = (\omega_1 + \omega_2)/2$, the MATLAB interp1 function was used to interpolate the solution from the previous mode superposition solution.

```
At Omega = 8.78 rad/s, |U| = 0.004432, 0.005524, 0.004314
```

Part F

MATLAB was used to solve for the response of the system to the sinusoidal loading. The process and results are described below.

Solving the eigensystem,

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}$$

After finding the eigenvalues, the natural frequency of the r-th mode is given by

$$\omega_r = \sqrt{\lambda_r}$$

Generalized mass, stiffness, and damping matrices are calculated as follows:

$$\mathcal{M} = \mathbf{\Phi}^\top \mathbf{M\Phi}$$

$$\mathcal{K} = \mathbf{\Phi}^\top \mathbf{K\Phi}$$

$$\mathcal{C} = \mathbf{\Phi}^\top \mathbf{C\Phi} = \text{diag}(2\zeta_r \omega_r \mathcal{M}_r)$$

Now, the equation of motion can be written in modal coordinates as

$$\mathcal{M}\ddot{\mathbf{q}} + \mathcal{C}\dot{\mathbf{q}} + \mathcal{K}\mathbf{q} = \mathbf{\Phi}^\top \mathbf{p}(t)$$

Again, since this is a 3-DOF system, we now have 3 uncoupled ODEs in modal coordinates.

```
First uncoupled differential equation:
2.942 q'' + 3.324 q' + 93.92 q = 0.577*cos(Omega*t)

Second uncoupled differential equation:
1.529 q'' + 7.280 q' + 216.59 q = 0.744*cos(Omega*t)

Third uncoupled differential equation:
1.322 q'' + 13.193 q' + 365.59 q = -0.833*cos(Omega*t)
```

To plot the responses, a value for the forcing frequency was assumed (i.e., $\Omega = 2$ rad/s). Solutions to ODEs:

```
u_1(t) =
0.004647*cos(2.0*t) + 0.0003861*sin(2.0*t) + ...
    6.852e-5*exp(-4.988*t)*sin(15.86*t) + ...
    0.0007135*exp(-2.38*t)*cos(11.66*t) - ...
    0.005573*exp(-0.565*t)*cos(5.622*t) + ...
    0.0001541*exp(-2.38*t)*sin(11.66*t) - ...
    0.0007205*exp(-0.565*t)*sin(5.622*t) + ...
    0.0002117*exp(-4.988*t)*cos(15.86*t)

u_2(t) =
0.008561*cos(2.0*t) + 0.0006474*sin(2.0*t) - ...
    0.0006206*exp(-4.988*t)*sin(15.86*t) - ...
    0.002615*exp(-2.38*t)*cos(11.66*t) - ...
    0.004029*exp(-0.565*t)*cos(5.622*t) - ...
    0.0005648*exp(-2.38*t)*sin(11.66*t) - ...
    0.0005209*exp(-0.565*t)*sin(5.622*t) - ...
    0.001917*exp(-4.988*t)*cos(15.86*t)

u_3(t) =
0.002185*cos(2.0*t) + 0.0001601*sin(2.0*t) + ...
    0.0004059*exp(-4.988*t)*sin(15.86*t) - ...
    0.00224*exp(-2.38*t)*cos(11.66*t) - ...
    0.001198*exp(-0.565*t)*cos(5.622*t) - ...
    0.0004838*exp(-2.38*t)*sin(11.66*t) - ...
    0.000155*exp(-0.565*t)*sin(5.622*t) + ...
    0.001254*exp(-4.988*t)*cos(15.86*t)
```

The responses can be seen in Figure 12.

Comparing the plots to the results in Part E, the amplitudes look comparable with the exception of the third degree of freedom.

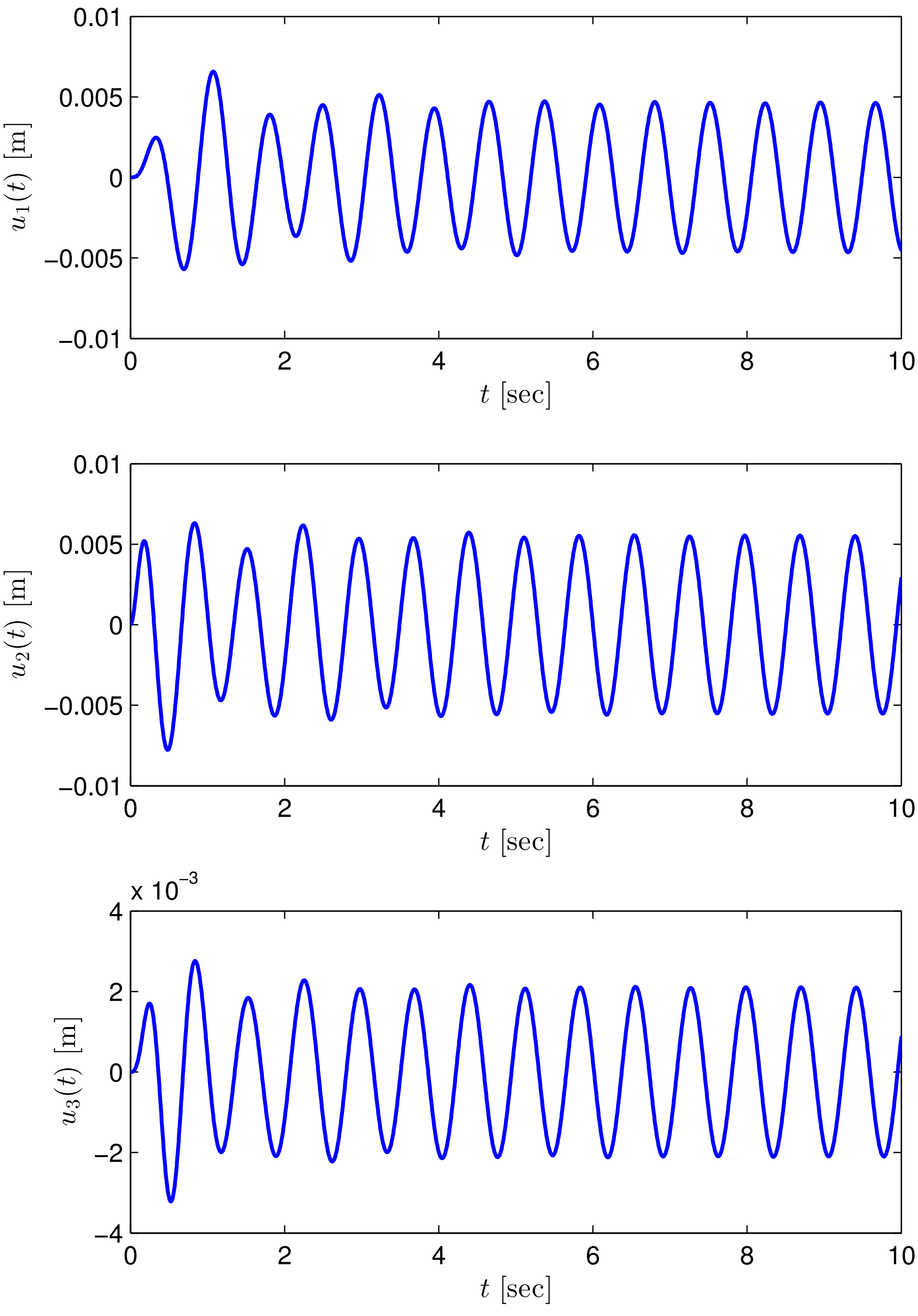


Figure 12. Response of proportionally damped system to sinusoidal loading.

Problem 6

Problem Statement

For a supported-clamped circular steel beam with a length of 2 meters and total mass of 50 kg, find the natural frequency of the first mode. Choose your own E and ρ for the material. Cite the source of data.

(**a**). Use MASDAN analytical solution

(**b**). Use Ritz method (by DYSSOLVE).

  (a) What is the lowest order polynomial (NP) required to solve the problem?

  (b) Find 3 solutions by DYSSOLVE using polynomial of order NP, NP + 1, and NP + 2.

(**c**). Use finite element method by DYSSOLVE. Model the beam using 1, 2, and 20 elements.

Tabulate your solutions.

Solution

The density and Modulus of Elasticity were determined from Hibbeler's Mechanics of Materials textbook assuming A36 steel.

$$\rho = 7.85 \text{ Mg/m}^3 \quad \text{and} \quad E = 200 \text{ GPa}$$

Part A

$$\boxed{\omega_1 = 9.79 \text{ rad/s}}$$

Table 1. Summary of results.

Method	Model 1	Model 2	Model 3
Ritz	9.82	9.80	9.79
FE	13.02	9.89	9.79

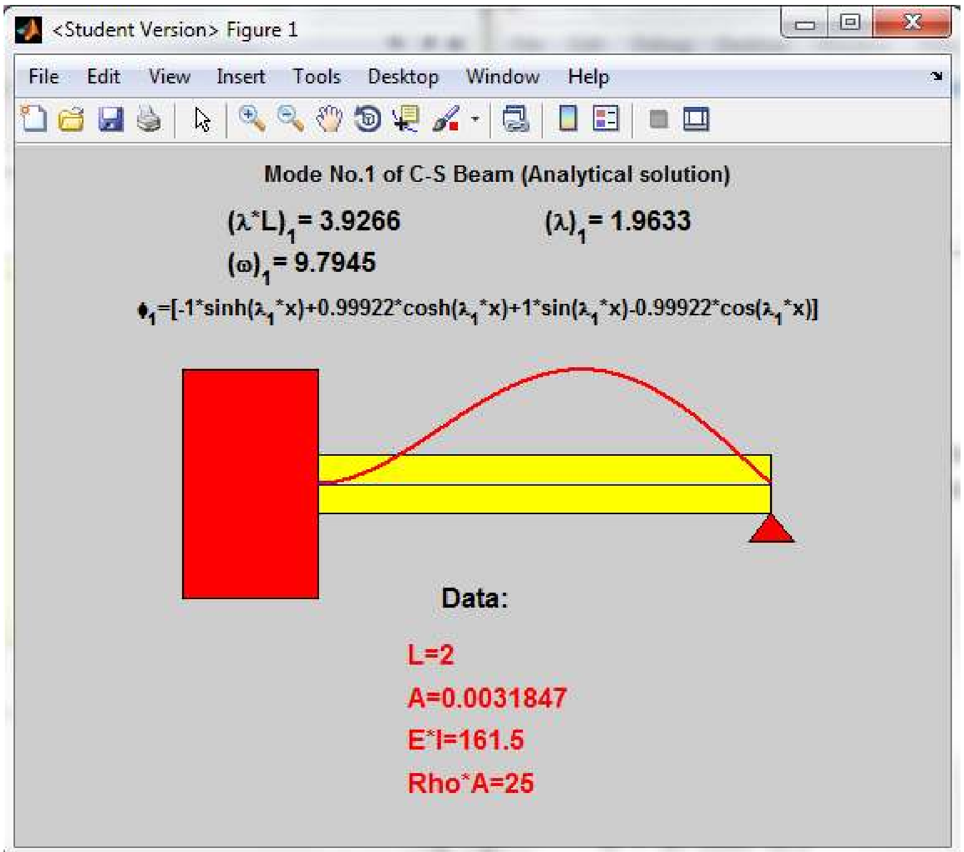


Figure 13. Analytical solution by MASDAN.

Part B

The lowest order polynomial required to solve the problem is 4.

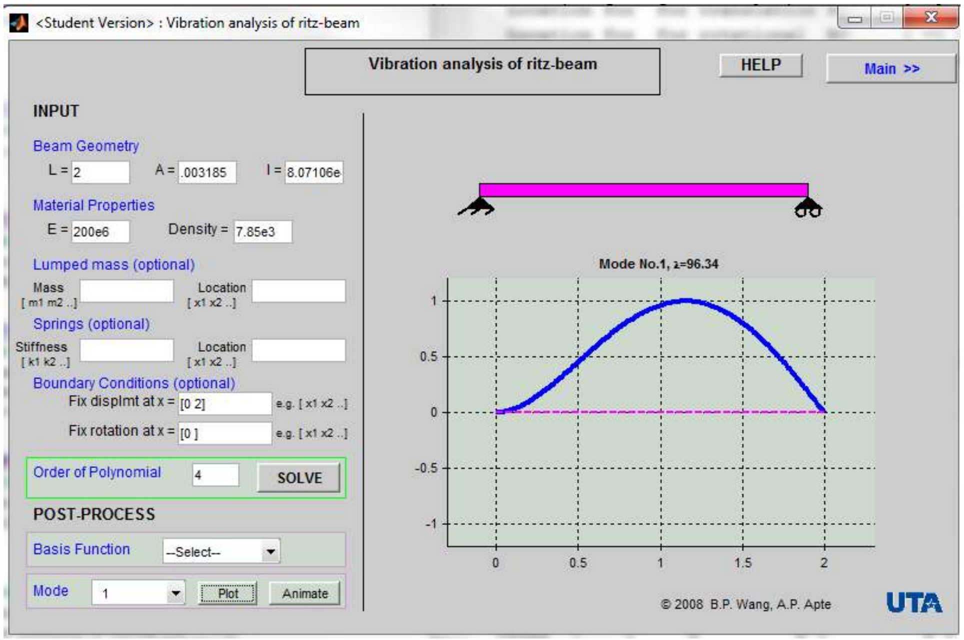


Figure 14. Ritz solution with NP = 4.

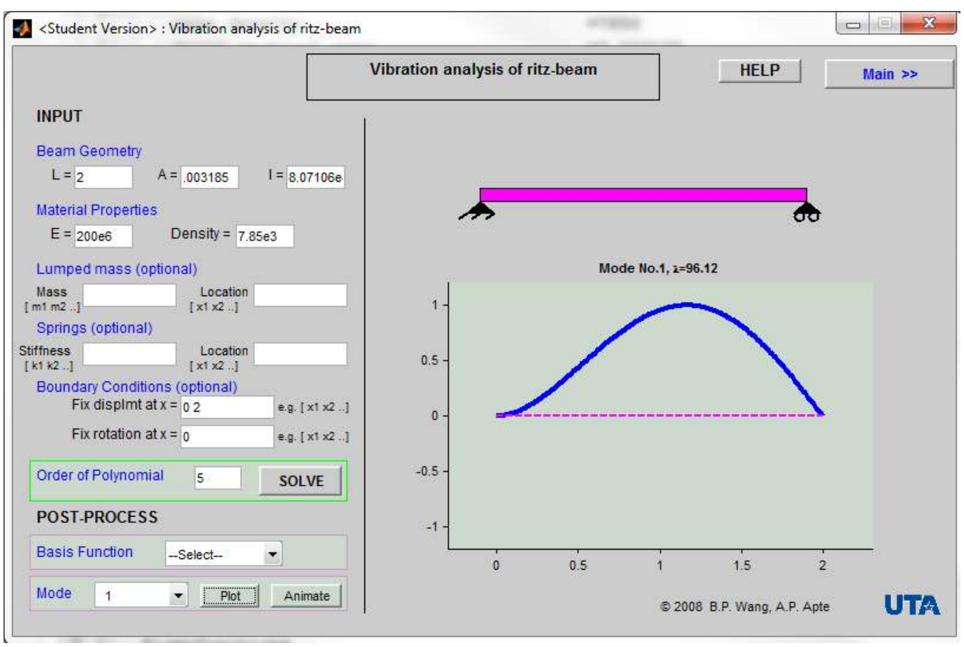


Figure 15. Ritz solution with NP = 5.

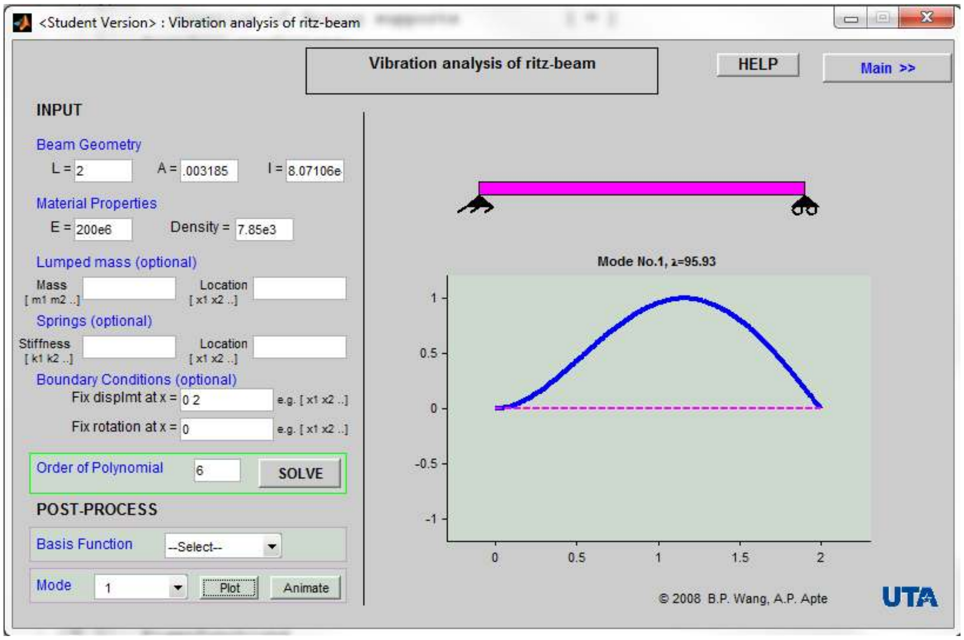


Figure 16. Ritz solution with NP = 6.

Part C

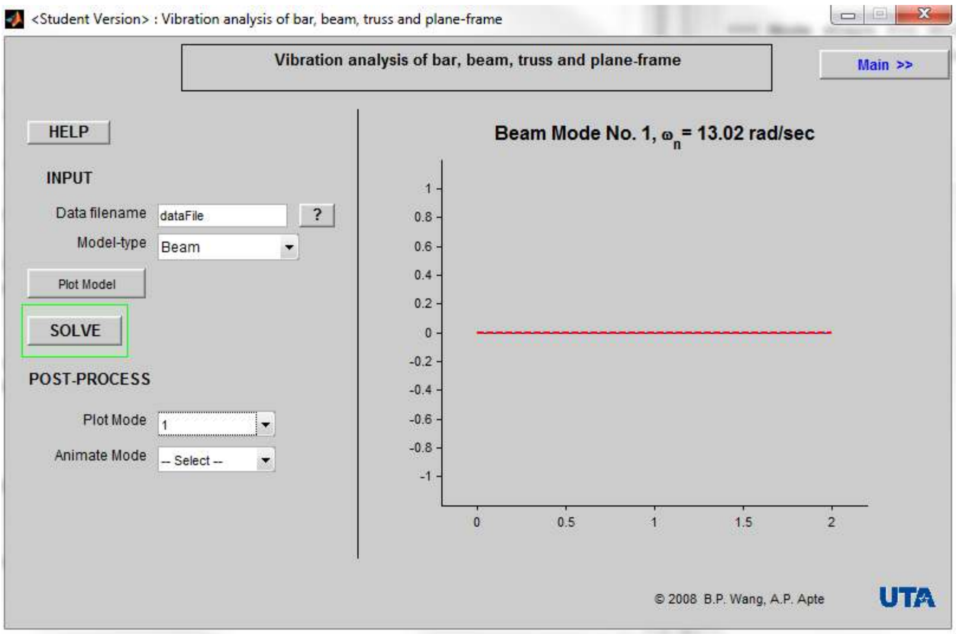


Figure 17. FEM solution with 1 element.

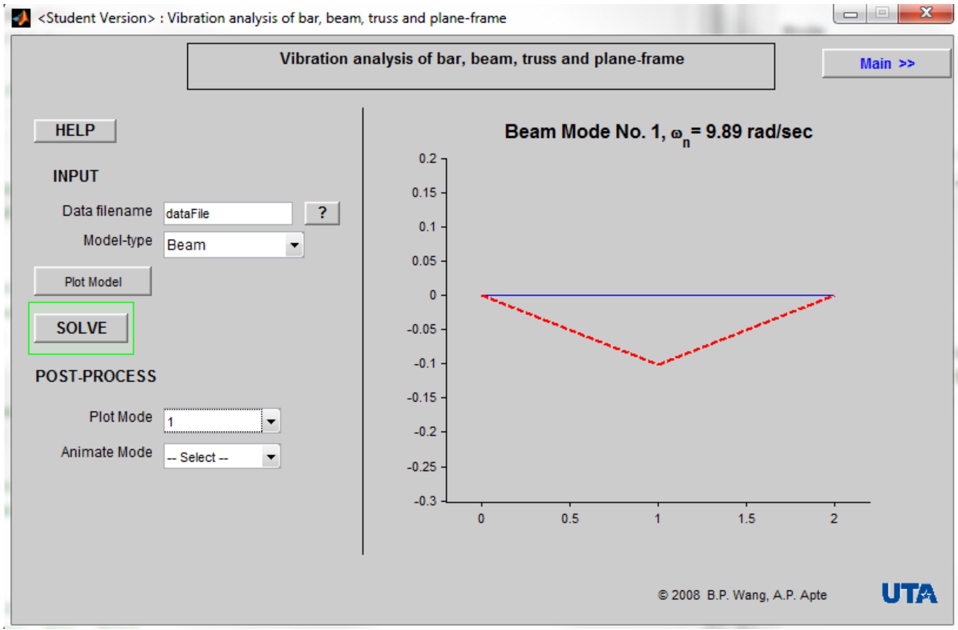


Figure 18. FEM solution with 2 elements.

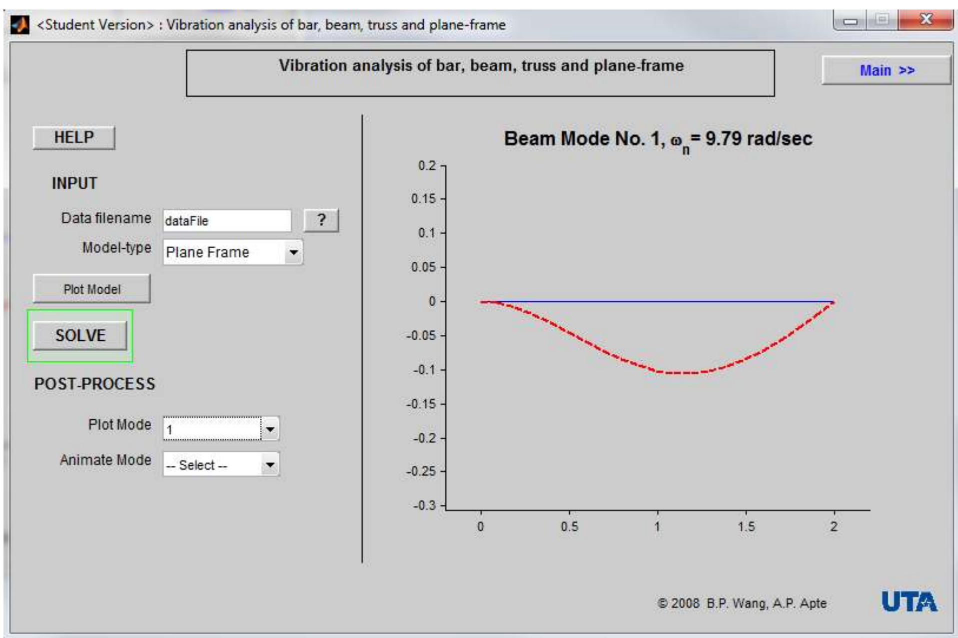


Figure 19. FEM solution with 20 elements.

Problem 7

Problem Statement

A hinge is to be added to the spring supported rigid bar. Find the optimal location of the hinge to maximize the natural frequency of the system. Use the following numerical data:

$$m = 40 \text{ kg}, \quad L = 10 \text{ m}, \quad k = 10,000 \text{ N/m}, \quad k_1 = 2k, \quad k_2 = k$$

The bar has a uniform mass distribution.

Remarks: This problem was a problem in Exam 1. I do not want to see the same solution. However, using the knowledge you have learned from my recent lectures, you can find the best support location from the mode shape plot of the 2-DOF bar without the hinge. Try to solve the problem using this approach.

Solution

The Assumed Modes method will be used to re-solve this problem. The calculations for this problem were performed using Mathematica. The code is at the end of this document. Using Note-19, the mass and stiffness matrices for this system are

$$\mathbf{M} = \begin{bmatrix} m & Lm/2 \\ Lm/2 & L^2m/3 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & Lk_2 \\ Lk_2 & L^2k_2 \end{bmatrix}$$

Next, the eigenvalues and eigenvectors are found.

$$\phi_1 = \begin{Bmatrix} 0.985329 \\ 0.170664 \end{Bmatrix}, \quad \phi_2 = \begin{Bmatrix} 0.985329 \\ -0.170664 \end{Bmatrix} \quad \text{and} \quad \lambda_{2,1} = \{2366.03, 633.975\}$$

Next, the shape function is defined.

$$F_x = \{x^0, x\}$$

$$\psi_r(x) = F_x c_r \quad \text{where} \quad c_r = \phi_r$$

Therefore, the two eigenfunctions are

$$\begin{aligned} \psi_1 &= F_x\phi_1 = 0.170664x + 0.985329 \\ \psi_2 &= F_x\phi_2 = 0.985329 - 0.170664x \end{aligned}$$

Plotting with Mathematica,

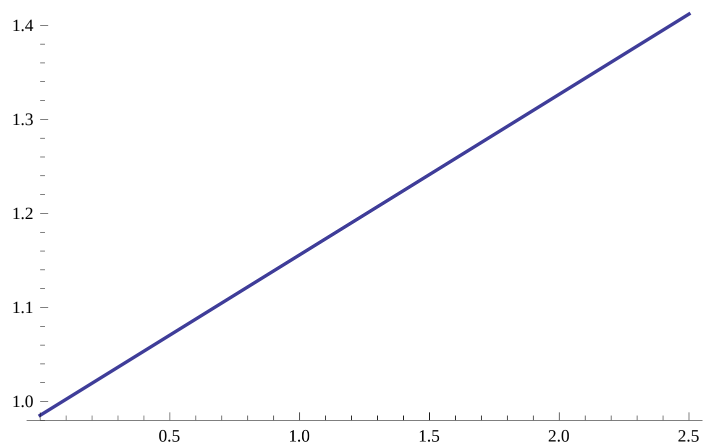


Figure 20. First eigenfunction.

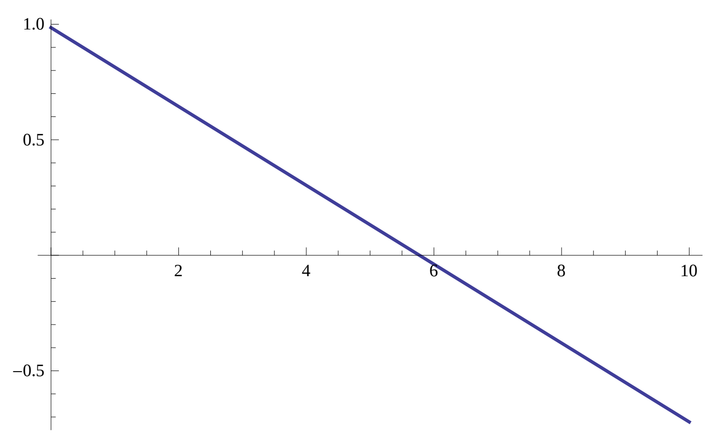


Figure 21. Second eigenfunction.

Examining Figure 21, it is evident that there is a node. The eigenfunction is solved for the node, yielding:

$$\boxed{x_{\text{optimal}} = 5.7735 \text{ m}}$$

Final Exam MATLAB code

Problem 2 MATLAB code

```
%% Clearing workspace
clc,clear,close all

%% Inputs

% Setting defaults
set(0,'DefaultTextInterpreter','LaTeX')

% Path to images
ImgPath = 'C:\Users\James\Desktop\School\Courses\UTA\AE 5311 - Structural ...
    Dynamics\Images\';

P_s = 5;
k = 32;
m = 2;
omega_n = sqrt(k/m);

%% Calculations

% Time vector
t = linspace(0,10,1e3);

% Displacement eqn
u = P_s/k*(sin(omega_n*t) - omega_n*t.*cos(omega_n*t));

% Plotting
figure('Position',[1010 60 560 250])
plot(t,u,'-k','LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u(t)$ [m]','FontSize',11)

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb2a.eps'])
```

Problem 3 MATLAB code

```
%% Clearing workspace
clc,clear,close all

%% Inputs

% Setting defaults
set(0,'DefaultTextInterpreter','LaTeX')

% Path to images
ImgPath = 'C:\Users\James\Desktop\School\Courses\UTA\AE 5311 - Structural ...
    Dynamics\Images\';

m = 2;                                  % kg
k = 32;                                 % N/m
T_0 = 2;                                % sec
P_0 = 64;                               % N
omega_n = sqrt(k/m);

%% Calculations

% Time vector
t_final = 10;
t = linspace(0,t_final,1e3);
sum = 0;

% Number of terms
nTerms = 15;

% Preallocating b vector for terms
b = zeros(1,nTerms);
Omega = b;

% Setting up string of linestyles for plotting of Fourier series
cString = {'-k','-b','-r','-g','--k','--b','--r','--g'};
counter = 1;

% Creating figure
% figure('Position',[1010 50 560 280])
% hold on

% Summing terms
for n = 1:nTerms

    % Sequence for odd numbers
    k = 2*n-1;
```

```
    fprintf('k = %d\n',k)

    % Logic structure to change sign of b_n term
    if k == 4*counter - 3
        b(n) = 4*P_0*T_0/(pi^2*k^2);
        counter = counter+1;
    else
        b(n) = -4*P_0*T_0/(pi^2*k^2);
    end

    % Summing terms
    sum = sum + b(n)*sin(2*k*pi*t/T_0);

    % Forcing frequency
    Omega(n) = 2*k*pi/T_0;

end

% Setting up figure to plot triangular wave
figure('Position',[1010 50 560 280])
plot(t,sum,'-k','LineWidth',1.5)
grid on

% Adding labels
xlabel('$t$ [sec]','FontSize',11)
ylabel('$f(t)$ [N]','FontSize',11)

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb3trianglewave.eps'])

%% Solving for u(t)

% Assigning A_i
A = b;

% Initializing reponse variable
u = 0;

% Summing first five terms of response
for n = 1:nTerms

    % Equation for response
    u = u + A(n)/k*omega_n/(omega_n^2 - Omega(n)^2)*...
        (omega_n*sin(Omega(n)*t) - Omega(n)*sin(omega_n*t));

end

% Plotting response
```

```
figure('Position',[1010 50 560 280])
plot(t,u,'-k','LineWidth',1.5)
grid

% Adding labels
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u(t)$ [m]','FontSize',11)

%% Solving for the response at t = 4

% Time at which we want to know the response
tr = 4;

% Interpolating from the previous calculation
ur = interp1(t,u,tr);

% Displaying results
fprintf('At t = %2.2f, u(%2.2f) = %2.2f m\n\n',tr,tr,ur)

% Plotting point on response
hold on
plot(tr,ur,'ro','LineWidth',1.5)

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb3response.eps'])
```

```
function [u,tf] = FinalExamProb4Validation(P_0,T_0,k,m)

nPoints = 1e3;
omega_n = sqrt(k/m);
tSPAN = linspace(0,T_0,nPoints);
IC = [0; 0];
[tf,u] = ode45(@func,tSPAN,IC);

%% Function to create triangle wave

    function Twave = triangleWave(t)

        % Preallocating Twave
        Twave = zeros(1,numel(t));

        % Setting up loop
        for n = 1:numel(t)

            if t(n) <= T_0/4
                Twave(n) = 4*P_0/T_0*t(n);
```

```
            elseif t(n) > T_0/4 && t(n) <= 3*T_0/4
                Twave(n) = -4*P_0/T_0*t(n) + 2*P_0;
            elseif t(n) > 3*T_0/4 && t(n) <= 5*T_0/4
                Twave(n) = 4*P_0/T_0*t(n) - 4*P_0;
            else
                break
            end

        end
    end

%% Alternate function for triangle wave


%% Function for ode45

    function dx = func(t,x)
        dx = zeros(2,1);
        dx(1) = x(2);
        dx(2) = triangleWave(t)/m - omega_n^2*x(1);
    end

end
```

Problem 4 MATLAB code

```
%% Clearing workspace
clc,clear,close all

%% Inputs

% Setting defaults
set(0,'DefaultTextInterpreter','LaTeX')

% Path to images
ImgPath = ['C:\Users\James\Desktop\School\Courses\UTA\AE 5311 - ',...
    'Structural Dynamics\Images\'];

m = 2;                                  % kg
k = 32;                                 % N/m
T_0 = 2;                                % sec
P_0 = 64;                               % N
omega_n = sqrt(k/m);

%% Calculations for Problem 3
```

```
% Time vector
t_final = T_0;
t = linspace(0,t_final,5e3);
sum = 0;

% Number of terms
nTerms = 16;

% Preallocating b vector for terms
b = zeros(1,nTerms);
Omega = b;

% Setting up string of linestyles for plotting of Fourier series
cString = {'-k','-b','-r','-g','--k','--b','--r','--g'};
counter = 1;

% Summing terms
for n = 1:nTerms

    % Sequence for odd numbers
    k = 2*n-1;
    fprintf('k = %d\n',k)

    % Logic structure to change sign of b_n term
    if k == 4*counter - 3
        b(n) = 4*P_0*T_0/(pi^2*k^2);
        counter = counter+1;
    else
        b(n) = -4*P_0*T_0/(pi^2*k^2);
    end

    % Summing terms
    sum = sum + b(n)*sin(2*k*pi*t/T_0);

    % Forcing frequency
    Omega(n) = 2*k*pi/T_0;

end

% Creating figure
figure('Position',[1010 50 560 280])
plot(t,sum,'-k','LineWidth',1.5)

% Solving for u(t)

% Assigning A_i
A = b;
term = zeros(numel(A),numel(t));
```

```
% Initializing reponse variable
u = 0;

% Summing first five terms of response
for n = 1:nTerms

    % Calculating each term
    term(n,:) = A(n)/k*omega_n/(omega_n^2 - Omega(n)^2)*...
        (omega_n*sin(Omega(n)*t) - Omega(n)*sin(omega_n*t));

    % Equation for response
    u = u + A(n)/k*omega_n/(omega_n^2 - Omega(n)^2)*...
        (omega_n*sin(Omega(n)*t) - Omega(n)*sin(omega_n*t));

end

% Plotting response
figure('Position',[1010 50 560 280])
hfourier = plot(t,u,'-b','LineWidth',1.5);
hold on

% Adding labels
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u(t)$ [m]','FontSize',11)

%% Calculations for Problem 4

% Inputs
m = 2;                                   % kg
k = 32;                                  % N/m
T_0 = 2;                                 % sec
P_0 = 64;                                % N
omega_n = sqrt(k/m);
T_1 = T_0/4;
t_final = 10;

% Time vectors
t_1 = linspace(0,T_1,1e3);
t_2 = linspace(T_1,3*T_1,1e3);
t_3 = linspace(3*T_1,4*T_1,1e3);
t_4 = linspace(4*T_1,t_final,1e3);

% First interval
u_1 = P_0/(m*T_1*omega_n^3)*(t_1*omega_n - sin(omega_n*t_1));

% Plotting
plot(t_1,u_1,'-r','LineWidth',1.5)

```

```
% Second interval
u_2 = -P_0/(m*T_1*omega_n^3)*(sin(omega_n*t_2) - 2*sin(omega_n*(t_2 - T_1)) ...
    + ...
    (t_2 - 2*T_1)*omega_n);

% Plotting
hold on
plot(t_2,u_2,'-r','LineWidth',1.5)

% Third interval
u_3 = -P_0/(m*T_1*omega_n^3)*(sin(omega_n*t_3) + 2*sin(omega_n*...
    (t_3 - 3*T_1)) - 2*sin(omega_n*(t_3-T_1)) - omega_n*(t_3 - 4*T_1));

hduhamel = plot(t_3,u_3,'-r','LineWidth',1.5);

% Fourth interval
u_4 = P_0/(m*T_1*omega_n^3)*(2*sin(omega_n*(t_4 - T_1))...
    - 2*sin(omega_n*(t_4 - 3*T_1))) + sin(omega_n*(t_4 - 4*T_1)) ...
    - sin(omega_n*t_4);


%% Calling ode45 for numerical solution
[u,t] = FinalExamProb4Validation(P_0,T_0,k,m);

% Separating solution
uNumerical = u(:,1);
hode45 = plot(t,uNumerical,'--k','LineWidth',1.5);

% Adding legend
lh = legend([hfourier,hduhamel,hode45],['Fourier Series w/ ',...
    num2str(nTerms),' terms'],'Duhamel Integral','ode45');
set(lh,'Location','SouthWest','FontSize',11,'Interpreter','LaTeX')

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4.eps'])

% Plotting
figure('Position',[1010 50 560 280])
plot(t_1,u_1,'-k',t_2,u_2,'--k',t_3,u_3,'-b',t_4,u_4,'--b',...
    'LineWidth',1.5)
xlabel('$t$ [sec]')
ylabel('$u(t)$ [m]')

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4d.eps'])
```

Problem 5 MATLAB code

```
%% Take Home Final Problem 5

%% Clearing workspace
clc,clear,close all

%% Inputs

% Setting defaults
set(0,'DefaultTextInterpreter','LaTeX')

% Path to images
ImgPath = ['C:\Users\James\Desktop\School\Courses\UTA\',...
    'AE 5311 - Structural Dynamics\Images\'];


K = [200 -100 0; -100 200 -100; 0 -100 400];
M = [4 0 0; 0 1 0; 0 0 2];
zeta = [0.1;0.2;0.3];
u_0 = [0;0;0];
v_0 = [0;1;0];
P = [0;0;0];

% Final time for plotting
t_final = 50;
nPoints = 1e4;

%% Part A - Find analytical solution to initial conditions

% Symbolically finding the eigenvalues
syms LL
eq1 = sort(real(vpa(solve(det(K - LL*M)==0,'LL'),4)));

% Using MATLAB function to check
[Phi,Lambda] = eig(K,M);

% Separating eigenvalues
Lambda = sort(diag(Lambda));
lambda_1 = Lambda(1);
lambda_2 = Lambda(2);
lambda_3 = Lambda(3);

% Finding eigenvectors
phi_1 = null(K - lambda_1*M);
phi_2 = null(K - lambda_2*M);
phi_3 = null(K - lambda_3*M);
```

```

% Natural frequencies
omega_1 = sqrt(lambda_1);
omega_2 = sqrt(lambda_2);
omega_3 = sqrt(lambda_3);

% Constructing Phi matrix
Phi = [phi_1 phi_2 phi_3];

% Generalized mass and stiffness matrices
GM = Phi'*M*Phi;
GK = Phi'*K*Phi;

% Generalized damping matrix
GC(1,1) = 2*zeta(1)*omega_1*GM(1,1);
GC(2,2) = 2*zeta(2)*omega_2*GM(2,2);
GC(3,3) = 2*zeta(3)*omega_3*GM(2,2);

disp('==================================================================')
disp('Part A - Modal EOMs')
disp('==================================================================')

% First uncoupled differential equation
term11 = GM(1,1);
term12 = 2*GM(1,1)*omega_1*zeta(1);
term13 = omega_1^2*GM(1,1);
term14 = phi_1'*P;
fprintf('First uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f \n\n',...
    term11,term12,term13,term14);

% Second uncoupled differential equation
term21 = GM(2,2);
term22 = 2*GM(2,2)*omega_2*zeta(2);
term23 = omega_2^2*GM(2,2);
term24 = phi_2'*P;
fprintf('Second uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f \n\n',...
    term21,term22,term23,term24);

% Third uncoupled differential equation
term31 = GM(3,3);
term32 = 2*GM(3,3)*omega_3*zeta(3);
term33 = omega_3^2*GM(3,3);
term34 = phi_3'*P;
fprintf('Third uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f \n\n',...
    term31,term32,term33,term34);

```

```
% Setting up derivatives
syms q1(t) q2(t) q3(t)
Dq1 = diff(q1);
D2q1 = diff(q1,2);
Dq2 = diff(q2);
D2q2 = diff(q2,2);
Dq3 = diff(q3);
D2q3 = diff(q3,2);

% Transforming initial conditions to modal coordinates
% Initial displacements
q1_0 = 1/GM(1,1)*phi_1'*M*u_0;
q2_0 = 1/GM(2,2)*phi_2'*M*u_0;
q3_0 = 1/GM(3,3)*phi_3'*M*u_0;

% Initial velocities
q1dot_0 = 1/GM(1,1)*phi_1'*M*v_0;
q2dot_0 = 1/GM(2,2)*phi_2'*M*v_0;
q3dot_0 = 1/GM(3,3)*phi_3'*M*v_0;

% Solving differential equations symbolically
Q1 = vpa(dsolve(term11*D2q1 + term12*Dq1 + term13*q1 == term14,...
    q1(0)==q1_0,Dq1(0)==q1dot_0),4);

Q2 = vpa(dsolve(term21*D2q2 + term22*Dq2 + term23*q2 == term24,...
    q2(0)==q2_0,Dq2(0)==q2dot_0),4);

Q3 = vpa(dsolve(term31*D2q3 + term32*Dq3 + term33*q3 == term34,...
    q3(0)==q3_0,Dq3(0)==q3dot_0),4);

% Substituting time vector
t_span = linspace(0,t_final,nPoints);

% Transforming from modal back to spatial coordinates
U = vpa(Phi*[Q1; Q2; Q3],4);

% Outputting solutions
disp('==================================================================')
disp('Solution for Part A')
disp('==================================================================')
disp('u_1(t) = ')
disp(vpa(simplify(U(1)),4))
disp('u_2(t) = ')
disp(vpa(simplify(U(2)),4))
disp('u_3(t) = ')
disp(vpa(simplify(U(3)),4))

% Substituting time vectors into symbolic expressions
u_1 = subs(U(1),t,t_span);
```

```
u_2 = subs(U(2),t,t_span);
u_3 = subs(U(3),t,t_span);
keepu_1 = u_1;

% Plotting response
figure('Position',[1060 40 530 775])

% Plot for first degree of freedom
subplot(3,1,1)
plot(t_span,u_1,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_1(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Plot for second degree of freedom
subplot(3,1,2)
plot(t_span,u_2,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_2(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Plot for third degree of freedom
subplot(3,1,3)
plot(t_span,u_3,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_3(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4a.eps'])

%% Part B

% Setting final time
% t_final = 50;

% Setting initial conditions equal to zero
u_0 = [0;0;0];
v_0 = [0;0;0];

% Assigning omega
Omega = 1/2*(omega_1 + omega_2);

% Forcing vector
P = [0;cos(Omega*t);0];

% Recalculating terms of the differential equation

```

```
term14 = phi_1'*P;
term24 = phi_2'*P;
term34 = phi_3'*P;

% Printing modal coordinates
disp('====================================================================')
disp('Part B - Modal EOMs')
disp('====================================================================')
fprintf('First uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f*cos(Omega*t) \n\n',...
    term11,term12,term13,phi_1'*[0;1;0]);
fprintf('Second uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f*cos(Omega*t) \n\n',...
    term21,term22,term23,phi_2'*[0;1;0]);
fprintf('Third uncoupled differential equation: \n')
fprintf('%1.3f q'''' + %1.3f q'' + %2.2f q = %1.3f*cos(Omega*t) \n\n',...
    term31,term32,term33,phi_3'*[0;1;0]);

% Initial displacements
q1_0 = 1/GM(1,1)*phi_1'*M*u_0;
q2_0 = 1/GM(2,2)*phi_2'*M*u_0;
q3_0 = 1/GM(3,3)*phi_3'*M*u_0;

% Initial velocities
q1dot_0 = 1/GM(1,1)*phi_1'*M*v_0;
q2dot_0 = 1/GM(2,2)*phi_2'*M*v_0;
q3dot_0 = 1/GM(3,3)*phi_3'*M*v_0;

% Solving differential equations
Q1 = vpa(simple(dsolve(term11*D2q1 + term12*Dq1 + term13*q1 == term14,...
    q1(0)==q1_0,Dq1(0)==q1dot_0)),4);

Q2 = vpa(simple(dsolve(term21*D2q2 + term22*Dq2 + term23*q2 == term24,...
    q2(0)==q2_0,Dq2(0)==q2dot_0)),4);

Q3 = vpa(simple(dsolve(term31*D2q3 + term32*Dq3 + term33*q3 == term34,...
    q3(0)==q3_0,Dq3(0)==q3dot_0)),4);

% Substituting time vector
t_span = linspace(0,t_final,nPoints);

% Transforming from modal back to spatial coordinates
U = vpa(Phi*[Q1; Q2; Q3],4);
disp('====================================================================')
disp('Solution for Part B')
disp('====================================================================')
disp('u_1(t) = ')
disp(vpa(simplify(U(1)),4))
disp('u_2(t) = ')
```

```
disp(vpa(simplify(U(2)),4))
disp('u_3(t) = ')
disp(vpa(simplify(U(3)),4))
u_1 = subs(U(1),t,t_span);
u_2 = subs(U(2),t,t_span);
u_3 = subs(U(3),t,t_span);

% Plotting response
figure('Position',[1060 40 530 775])

% Plot for first degree of freedom
subplot(3,1,1)
plot(t_span,u_1,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_1(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Plot for second degree of freedom
subplot(3,1,2)
plot(t_span,u_2,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_2(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Plot for third degree of freedom
subplot(3,1,3)
plot(t_span,u_3,'LineWidth',1.5)
xlabel('$t$ [sec]','FontSize',11)
ylabel('$u_3(t)$ [m]','FontSize',11)
set(gca,'XLim',[0 10])

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4b.eps'])

%-------------------------------------------------------------------------
% Finding the frequency response function
%-------------------------------------------------------------------------

Omega_final = 50;
Omega = linspace(0,Omega_final,nPoints);

% Setting number of degrees of freedom
N = 3;

% Preallocating
r = zeros(N,nPoints);
alpha = r;
H_12 = zeros(1,nPoints);
```

```
H_22 = H_12;
H_32 = H_12;

% Calculating r and phase
for n = 1:N
    r(n,:) = Omega/sqrt(Lambda(n));
    alpha(n,:) = atan2(-2*zeta(n)*r(n,:),(1 - r(n,:).^2));
end

% Calculating numerical value for FRF
for n = 1:N
    H_12 = H_12 + Phi(1,n)*Phi(1,n)/GK(n,n)*1./((1 - r(n,:).^2) + ...
        1i*(2*zeta(n)*r(n,:)));
    H_22 = H_22 + Phi(2,n)*Phi(2,n)/GK(n,n)*1./((1 - r(n,:).^2) + ...
        1i*(2*zeta(n)*r(n,:)));
    H_32 = H_32 + Phi(3,n)*Phi(3,n)/GK(n,n)*1./((1 - r(n,:).^2) + ...
        1i*(2*zeta(n)*r(n,:)));
end

% Output the FRF at DOF 2
disp('===================================================================')
disp('Frequency Response Function for DOF 2')
disp('===================================================================')
syms W

% Calculating symbolic FRF to output
H1 = Phi(2,1)*Phi(2,1)/GK(1,1)*1/(1 - W^2/sqrt(Lambda(1))^2 + ...
    1i*(2*zeta(1)*W/sqrt(Lambda(1))));
H2 = Phi(2,2)*Phi(2,2)/GK(2,2)*1/(1 - W^2/sqrt(Lambda(2))^2 + ...
    1i*(2*zeta(2)*W/sqrt(Lambda(2))));
H3 = Phi(2,3)*Phi(2,3)/GK(3,3)*1/(1 - W^2/sqrt(Lambda(3))^2 + ...
    1i*(2*zeta(3)*W/sqrt(Lambda(3))));

% Outputting results
disp('N_1/D_1(W) = ')
vpa(H1,4)
disp('N_2/D_2(W) = ')
vpa(H2,4)
disp('N_3/D_3(W) = ')
vpa(H3,4)

% Plotting
figure('Position',[980 170 560 620])
subplot(2,1,1)
plot(r(2,:),abs(H_22),'-b','LineWidth',1.5)
set(gca,'YScale','log')
xlabel('$\Omega/\omega_2$ [rad/s]','FontSize',11)
ylabel('$|H_{22}(\Omega)|$','FontSize',11)

```

```
subplot(2,1,2)
plot(r(2,:),180/pi*alpha(2,:),'-b','LineWidth',1.5)
xlabel('$\Omega/\omega_2$ [rad/s]','FontSize',11)
ylabel('Phase angle $\alpha$ [deg]')

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4bFRF.eps'])

%% Part C - using fft to find U_1(Omega)

% Determining time step
Deltat = t_span(2) - t_span(1);

% Sampling frequency
f_s = 1/Deltat;

% Frequency resolution
Deltaf = 1/t_final;

% Number of samples
u_1 = keepu_1;
nfft = pow2(nextpow2(length(u_1)));

% Taking the fft
U_1 = fft(u_1,nfft);
F_1 = (0:nfft-1)*f_s/nfft;
p_1 = U_1.*conj(U_1)/nfft;

% Phase angle
phase = unwrap(angle(U_1));

% Creating the CO-QUAD plot
figure('Position',[935 300 560 300])
plot(F_1,real(U_1),'-k',F_1,imag(U_1),'-b','LineWidth',1.5)
xlabel('f [Hz]','FontSize',11)
set(gca,'YScale','log')
lh = legend('Real','Imaginary');
set(lh,'FontSize',11)

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4cCOQUAD.eps'])

% Creating Nyquist plot
figure('Position',[1150 80 430 360])
plot(real(U_1),imag(U_1),'LineWidth',1.5)
xlabel('$\Re$','FontSize',11)
ylabel('$\Im$','FontSize',11)
```

```

% Saving plot
set(gcf,'PaperPositionMode','auto')
print(gcf, '-depsc', [ImgPath,'FinalProb4cNyquist.eps'])

%% Part E - Finding the amplitude of the response at Omega

% Inputs
Omega = (omega_1 + omega_2)/2;

% Interpolating from previously calculated H values to find amplitude
U1 = interp1(r(1,:),abs(H_12),Omega/omega_1);
U2 = interp1(r(2,:),abs(H_22),Omega/omega_2);
U3 = interp1(r(3,:),abs(H_32),Omega/omega_3);

% Outputting results
fprintf('At Omega = %2.2f rad/s, |U| = %f, %f, %f\n\n',Omega,U1,U2,U3)
```

Final Exam Mathematica code

To expedite the problem solving process, some of the symbolic calculations were accomplished using Wolfram Mathematica. The files are below.

Problem 1

Part B

Clearing variables

```
In[17]:= ClearAll["Global`*"]
```

Defining the EOM

```
de = m*u''[t] + c*u'[t] + k*u[t]  == P0
```

$k\,u[t] + c\,u'[t] + m\,u''[t] == P_0$

Taking the Laplace Transform

```
LP = LaplaceTransform[de, t, s]
```

$$k\,\text{LaplaceTransform}[u[t], t, s] + c\,(s\,\text{LaplaceTransform}[u[t], t, s] - u[0]) + m\left(s^2\,\text{LaplaceTransform}[u[t], t, s] - s\,u[0] - u'[0]\right) == \frac{P_0}{s}$$

$$\mathbf{k\,LaplaceTransform[u[t], t, s] + c\,(s\,LaplaceTransform[u[t], t, s] - u[0]) + m\left(s^2\,LaplaceTransform[u[t], t, s] - s\,u[0] - u'[0]\right) == \frac{P_0}{s}}$$

$$k\,\text{LaplaceTransform}[u[t], t, s] + c\,(s\,\text{LaplaceTransform}[u[t], t, s] - u[0]) + m\left(s^2\,\text{LaplaceTransform}[u[t], t, s] - s\,u[0] - u'[0]\right) == \frac{P_0}{s}$$

Solving for U(s)

```
U = Solve[LP, LaplaceTransform[u[t], t, s]]
```

$$\left\{\left\{\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0 + c\,s\,u[0] + m\,s^2\,u[0] + m\,s\,u'[0]}{s\left(k + c\,s + m\,s^2\right)}\right\}\right\}$$

Assigning zero initial conditions

```
UU = U[[1]] /. u[0] → 0
```

$$\left\{\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0 + m\,s\,u'[0]}{s\left(k + c\,s + m\,s^2\right)}\right\}$$

```
UUU = UU[[1]] /. u'[0] → 0
```

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0}{s\left(k + c\,s + m\,s^2\right)}$$

Manipulating

U1 = UUU /. c → 2 * ζ * ω_n * m

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0}{s\left(k + m\,s^2 + 2\,m\,s\,\zeta\,\omega_n\right)}$$

U2 = U1 /. m → k / ω_n^2

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0}{s\left(k + \frac{k\,s^2}{\omega_n^2} + \frac{2\,k\,s\,\zeta}{\omega_n}\right)}$$

Factoring out P_0/k

U3 = FactorTerms[U2, P_0 / k]

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0}{s\left(k + \frac{k\,s^2}{\omega_n^2} + \frac{2\,k\,s\,\zeta}{\omega_n}\right)}$$

U4 = Apart[U3]

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_0}{k\,s} + \frac{-s\,P_0 - 2\,\zeta\,P_0\,\omega_n}{k\left(s^2 + 2\,s\,\zeta\,\omega_n + \omega_n^2\right)}$$

Taking the inverse Laplace transform

u1 = InverseLaplaceTransform[U4, s, t]

$$u[t] \to \frac{P_0}{k} - \frac{1}{2\,k\left(-1+\zeta^2\right)\omega_n^2} P_0 \sqrt{\left(-1+\zeta^2\right)\omega_n^2} \left(-e^{t\left(-\zeta\,\omega_n - \sqrt{(-1+\zeta^2)\,\omega_n^2}\right)}\,\zeta\,\omega_n + e^{t\left(-\zeta\,\omega_n + \sqrt{(-1+\zeta^2)\,\omega_n^2}\right)}\,\zeta\,\omega_n + e^{t\left(-\zeta\,\omega_n - \sqrt{(-1+\zeta^2)\,\omega_n^2}\right)} \sqrt{\left(-1+\zeta^2\right)\omega_n^2} + e^{t\left(-\zeta\,\omega_n + \sqrt{(-1+\zeta^2)\,\omega_n^2}\right)} \sqrt{\left(-1+\zeta^2\right)\omega_n^2}\right)$$

Simplify using zeta < 1

u2 = Refine[u1, ζ < 1 && ζ > 0]

$$u[t] \to \frac{P_0}{k} - \frac{1}{2\,k\left(-1+\zeta^2\right)\omega_n^2} \sqrt{1-\zeta^2}\; P_0 \sqrt{-\omega_n^2} \left(-e^{t\left(-\zeta\,\omega_n - \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2}\right)}\,\zeta\,\omega_n + e^{t\left(-\zeta\,\omega_n + \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2}\right)}\,\zeta\,\omega_n + e^{t\left(-\zeta\,\omega_n - \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2}\right)} \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2} + e^{t\left(-\zeta\,\omega_n + \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2}\right)} \sqrt{1-\zeta^2}\,\sqrt{-\omega_n^2}\right)$$

Simplifying using omega > 0

u3 = Refine[u2, ω_n > 0]

$$u[t] \to \frac{P_0}{k} - \frac{1}{2\,k\left(-1+\zeta^2\right)\omega_n}\, i \sqrt{1-\zeta^2}\; P_0 \left(-e^{t\left(-\zeta\,\omega_n - i\sqrt{1-\zeta^2}\,\omega_n\right)}\,\zeta\,\omega_n + e^{t\left(-\zeta\,\omega_n + i\sqrt{1-\zeta^2}\,\omega_n\right)}\,\zeta\,\omega_n + i\,e^{t\left(-\zeta\,\omega_n - i\sqrt{1-\zeta^2}\,\omega_n\right)} \sqrt{1-\zeta^2}\,\omega_n + i\,e^{t\left(-\zeta\,\omega_n + i\sqrt{1-\zeta^2}\,\omega_n\right)} \sqrt{1-\zeta^2}\,\omega_n\right)$$

Plugging in for omega_d

```
u4 = u3 /. Sqrt[1 - ζ^2] * ωn → ωd
```

$$u[t] \to \frac{P_0}{k} - \frac{1}{2\,k\,(-1+\zeta^2)\,\omega_n}\, i\,\sqrt{1-\zeta^2}\,P_0 \left(i\,e^{t\left(-\zeta\,\omega_n - i\sqrt{1-\zeta^2}\,\omega_n\right)}\,\omega_d + i\,e^{t\left(-\zeta\,\omega_n + i\sqrt{1-\zeta^2}\,\omega_n\right)}\,\omega_d - e^{t\,(-i\,\omega_d-\zeta\,\omega_n)}\,\zeta\,\omega_n + e^{t\,(i\,\omega_d-\zeta\,\omega_n)}\,\zeta\,\omega_n\right)$$

Attempt to express relationship in terms of trig functions

```
u5 = ExpToTrig[u4]
```

$$u[t] \to \frac{P_0}{k} - \frac{1}{2\,k\,(-1+\zeta^2)\,\omega_n}\, i\,\sqrt{1-\zeta^2}\,P_0 \Big(i\,\mathrm{Cosh}\big[t\,\zeta\,\omega_n - i\,t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d + i\,\mathrm{Cosh}\big[t\,\zeta\,\omega_n + i\,t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d - i\,\mathrm{Sinh}\big[t\,\zeta\,\omega_n - i\,t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d - i\,\mathrm{Sinh}\big[t\,\zeta\,\omega_n + i\,t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d - \zeta\,\mathrm{Cos}[t\,\omega_d - i\,t\,\zeta\,\omega_n]\,\omega_n + \zeta\,\mathrm{Cos}[t\,\omega_d + i\,t\,\zeta\,\omega_n]\,\omega_n + i\,\zeta\,\mathrm{Sin}[t\,\omega_d - i\,t\,\zeta\,\omega_n]\,\omega_n + i\,\zeta\,\mathrm{Sin}[t\,\omega_d + i\,t\,\zeta\,\omega_n]\,\omega_n\Big)$$

```
u6 = FullSimplify[u5]
```

$$u[t] \to \frac{1}{k\,(-1+\zeta^2)\,\omega_n}\, e^{-t\,\zeta\,\omega_n}\,P_0 \left(e^{t\,\zeta\,\omega_n}\,(-1+\zeta^2)\,\omega_n + \sqrt{1-\zeta^2}\left(\mathrm{Cos}\big[t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d + \zeta\,\mathrm{Sin}[t\,\omega_d]\,\omega_n\right)\right)$$

```
u7[t_] = u6
```

$$u[t] \to \frac{1}{k\,(-1+\zeta^2)\,\omega_n}\, e^{-t\,\zeta\,\omega_n}\,P_0 \left(e^{t\,\zeta\,\omega_n}\,(-1+\zeta^2)\,\omega_n + \sqrt{1-\zeta^2}\left(\mathrm{Cos}\big[t\sqrt{1-\zeta^2}\,\omega_n\big]\,\omega_d + \zeta\,\mathrm{Sin}[t\,\omega_d]\,\omega_n\right)\right)$$

Determining if this form is equal to book form

```
ubook1 = P0 / k * (1 - Exp[-ζ * ωn * t] * (Cos[ωd * t] + ζ * ωn / ωd * Sin[ωd * t]))
```

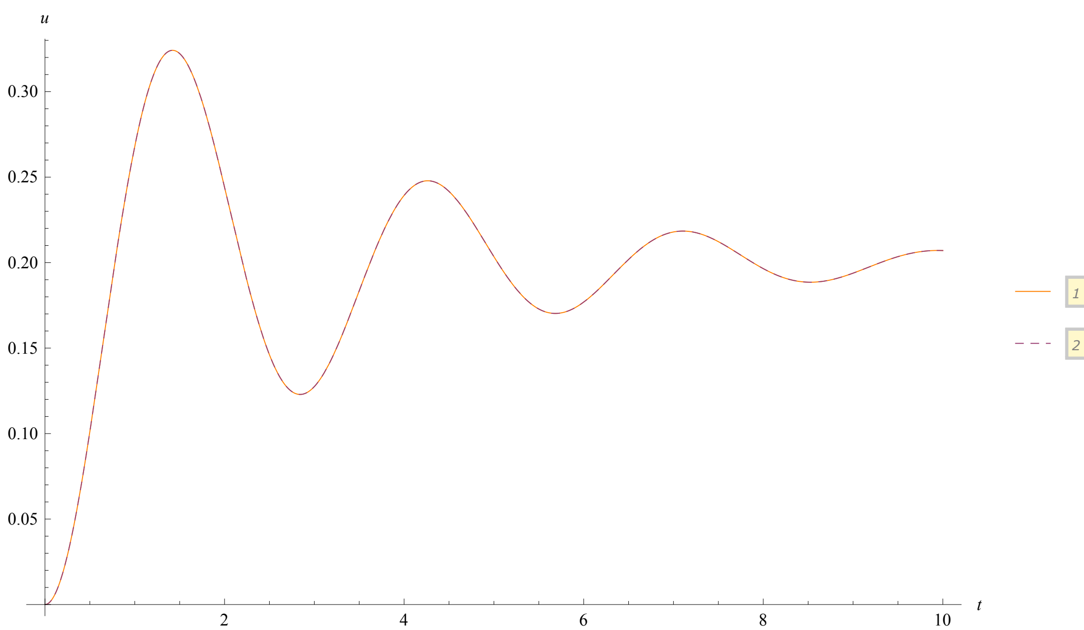
$$\frac{P_0\left(1 - e^{-t\,\zeta\,\omega_n}\left(\mathrm{Cos}[t\,\omega_d] + \frac{\zeta\,\mathrm{Sin}[t\,\omega_d]\,\omega_n}{\omega_d}\right)\right)}{k}$$

Testing equality by assigning values and plotting

```
P0 = 10;

ζ = 0.15;

ωn = Sqrt[5];

k = 50;

ωd = ωn * Sqrt[1 - ζ^2];
```

Creating plots

```
Plot[{Evaluate[ubook1], Evaluate[1/k 10 (1 - e^(-√5 t ξ) Cos[2.2107690969434146` t] -
      1/√(1 - ξ^2) e^(-√5 t ξ) ξ Sin[2.2107690969434146` t])]}, {t, 0, 10},
 Axes → True, AxesLabel → {t , u}, PlotStyle → {Orange, Dashed, Thick},
 PlotLegends → Automatic]
```



Part C

In this part, we are required to solve for the solution of an underdamped SDOF system subjected to a Sine input using the Laplace transform method. Assume zero initial conditions.

```
de = m*u''[t] + c*u'[t] + k*u[t]  == Ps Sin[Ωs*t]
```

$k\,u[t] + c\,u'[t] + m\,u''[t] == \text{Sin}[t\,\Omega_s]\,P_s$

Taking the Laplace transform

```
lt = LaplaceTransform[de, t, s]
```

$k\,\text{LaplaceTransform}[u[t], t, s] + c\,(s\,\text{LaplaceTransform}[u[t], t, s] - u[0]) + m\,(s^2\,\text{LaplaceTransform}[u[t], t, s] - s\,u[0] - u'[0]) == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$

Enforcing initial conditions

```
U1 = lt /. u[0] → 0
```

$$k\,\text{LaplaceTransform}[u[t], t, s] + c\,s\,\text{LaplaceTransform}[u[t], t, s] + m\left(s^2\,\text{LaplaceTransform}[u[t], t, s] - u'[0]\right) == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$$

```
U2 = U1 /. u'[0] → 0
```

$$k\,\text{LaplaceTransform}[u[t], t, s] + c\,s\,\text{LaplaceTransform}[u[t], t, s] + m\,s^2\,\text{LaplaceTransform}[u[t], t, s] == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$$

Solving for U(s)

```
U3 = Solve[U2, LaplaceTransform[u[t], t, s]]
```

$$\left\{\left\{\text{LaplaceTransform}[u[t], t, s] \to \frac{P_s\,\Omega_s}{\left(k + c\,s + m\,s^2\right)\left(s^2 + \Omega_s^2\right)}\right\}\right\}$$

```
U4 = U3[[1]];
```

```
U5 = U4[[1]];
```

Inserting relations for m, omega and zeta

```
U6 = U5 /. m → k / ωn^2
```

$$\text{LaplaceTransform}[u[t], t, s] \to \frac{P_s\,\Omega_s}{\left(k + c\,s + \frac{k\,s^2}{5}\right)\left(s^2 + \Omega_s^2\right)}$$

In[1]:=
```
ClearAll["Global`*"]
```

Part C

In this part, we are required to solve for the solution of an underdamped SDOF system subjected to a Sine input using the Laplace transform method. Assume zero initial conditions.

In[2]:=
```
de = m*u''[t] + c*u'[t] + k*u[t] == Ps Sin[Ωs*t]
```

Out[2]= $k\,u[t] + c\,u'[t] + m\,u''[t] == \mathrm{Sin}[t\,\Omega_s]\,P_s$

Taking the Laplace transform

In[3]:=

In[4]:=
```
lt = LaplaceTransform[de, t, s]
```

Out[4]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,(s\,\mathrm{LaplaceTransform}[u[t], t, s] - u[0]) + m\,(s^2\,\mathrm{LaplaceTransform}[u[t], t, s] - s\,u[0] - u'[0]) == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$

Enforcing initial conditions

In[5]:=
```
U1 = lt /. u[0] → 0
```

Out[5]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,s\,\mathrm{LaplaceTransform}[u[t], t, s] + m\,(s^2\,\mathrm{LaplaceTransform}[u[t], t, s] - u'[0]) == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$

In[6]:=
```
U2 = U1 /. u'[0] → 0
```

Out[6]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,s\,\mathrm{LaplaceTransform}[u[t], t, s] + m\,s^2\,\mathrm{LaplaceTransform}[u[t], t, s] == \frac{P_s\,\Omega_s}{s^2 + \Omega_s^2}$

Solving for U(s)

In[7]:=
```
U3 = Solve[U2, LaplaceTransform[u[t], t, s]]
```

Out[7]= $\left\{\left\{\mathrm{LaplaceTransform}[u[t], t, s] \to \frac{P_s\,\Omega_s}{(k + c\,s + m\,s^2)\,(s^2 + \Omega_s^2)}\right\}\right\}$

In[8]:=

In[9]:=
```
U3 = Flatten[U3];
```

Inserting relations for m, omega and zeta

In[10]:=
```
U6 = U5 /. c → 2*ξ*ωn*m
```

Out[10]= U5

In[11]:=
```
U7 = U6 /. m → k / ωn^2
```

Out[11]= U5

Expanding

In[12]:= **U8 = Apart[U7]**

Out[12]= U5

Attempting inverse Laplace

In[13]:= $$\mathbf{U9} = \frac{P_s\,\omega_n^2\,\Omega_s}{k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)}$$

Out[13]= $$\frac{P_s\,\omega_n^2\,\Omega_s}{k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)}$$

In[14]:= $$\mathbf{U[s\_]} = \frac{P_s\,\omega_n^2\,\Omega_s}{k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)}$$

Out[14]= $$\frac{P_s\,\omega_n^2\,\Omega_s}{k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)}$$

In[15]:= **Us1 = Apart[U[s], s]**

Out[15]= $$\frac{-2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s+P_s\,\omega_n^4\,\Omega_s-P_s\,\omega_n^2\,\Omega_s^3}{k\left(s^2+\Omega_s^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)}+\left(2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s-P_s\,\omega_n^4\,\Omega_s+4\,\zeta^2\,P_s\,\omega_n^4\,\Omega_s+P_s\,\omega_n^2\,\Omega_s^3\right)\Big/\left(k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)\right)$$

In[16]:= **Us2 = Refine[Us1, ζ > 0 && ζ < 1 && ω_n > 0]**

Out[16]= $$\frac{-2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s+P_s\,\omega_n^4\,\Omega_s-P_s\,\omega_n^2\,\Omega_s^3}{k\left(s^2+\Omega_s^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)}+\left(2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s-P_s\,\omega_n^4\,\Omega_s+4\,\zeta^2\,P_s\,\omega_n^4\,\Omega_s+P_s\,\omega_n^2\,\Omega_s^3\right)\Big/\left(k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)\right)$$

In[17]:= **Us3 = Refine[U[s], ω_n > 0 && P_s > 0 && k > 0 && Ω_s > 0]**

Out[17]= $$\frac{P_s\,\omega_n^2\,\Omega_s}{k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)}$$

In[18]:= **ExpandAll[Us3]**

Out[18]= $$\left(P_s\,\omega_n^2\,\Omega_s\right)\Big/\left(k\,s^4+2\,k\,s^3\,\zeta\,\omega_n+k\,s^2\,\omega_n^2+k\,s^2\,\Omega_s^2+2\,k\,s\,\zeta\,\omega_n\,\Omega_s^2+k\,\omega_n^2\,\Omega_s^2\right)$$

In[19]:= **Apart[Out[17], s]**

Out[19]= $$\frac{-2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s+P_s\,\omega_n^4\,\Omega_s-P_s\,\omega_n^2\,\Omega_s^3}{k\left(s^2+\Omega_s^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)}+\left(2\,s\,\zeta\,P_s\,\omega_n^3\,\Omega_s-P_s\,\omega_n^4\,\Omega_s+4\,\zeta^2\,P_s\,\omega_n^4\,\Omega_s+P_s\,\omega_n^2\,\Omega_s^3\right)\Big/\left(k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_s^2+4\,\zeta^2\,\omega_n^2\,\Omega_s^2+\Omega_s^4\right)\right)$$

Finding the poles

In[20]:= **D10 = Denominator[Us3]**

Out[20]= $k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)$

In[21]:= `D11 = Refine[D10, ζ > 0 && ζ < 1 && ωn > 0]`

Out[21]= $k\left(s^2+2\,s\,\zeta\,\omega_n+\omega_n^2\right)\left(s^2+\Omega_s^2\right)$

In[22]:= `D12 = Collect[D11, ωn]`

Out[22]= $k\,s^2\left(s^2+\Omega_s^2\right)+2\,k\,s\,\zeta\,\omega_n\left(s^2+\Omega_s^2\right)+k\,\omega_n^2\left(s^2+\Omega_s^2\right)$

Inserting solution to Partial fractions problem

In[23]:= `L1 = D12[[1]]`

Out[23]= $k\,s^2\left(s^2+\Omega_s^2\right)$

In[24]:= `L1 = L1[[1]]`

Out[24]= k

Substituting omega_d

In[25]:= `L1 = L1 /. I * Sqrt[ωn^2 - ζ * ωn^2] → I * ωd`

Out[25]= k

Poles

In[26]:= `Lam1 = 0;`

In[27]:= `Lam2 = -ζ * ωn - I * ωd`

Out[27]= $-\mathbb{i}\,\omega_d-\zeta\,\omega_n$

In[28]:= `Lam3 = -ζ * ωn + I * ωd`

Out[28]= $\mathbb{i}\,\omega_d-\zeta\,\omega_n$

Simplifying derived solution

In[29]:= `A1 = ωn^2 * Ωs * Ps / k / (2 * I * ωd * ((ζ * ωn + I * ωd)^2 + Ωs^2))`

Out[29]= $-\frac{\mathbb{i}\,P_s\,\omega_n^2\,\Omega_s}{2\,k\,\omega_d\left((\mathbb{i}\,\omega_d+\zeta\,\omega_n)^2+\Omega_s^2\right)}$

In[30]:= `B1 = ωn^2 * Ωs * Ps / k / (-2 * I * ωd * ((ζ * ωn - I * ωd)^2 + Ωs^2))`

Out[30]= $\frac{\mathbb{i}\,P_s\,\omega_n^2\,\Omega_s}{2\,k\,\omega_d\left((-\mathbb{i}\,\omega_d+\zeta\,\omega_n)^2+\Omega_s^2\right)}$

In[31]:= `C1 = ωn^2 * Ωs * Ps / k / (-2 * I * Ωs * ((-I * Ωs - ζ * ωn)^2 + ωd^2))`

Out[31]= $$\frac{i\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n - i\, \Omega_s)^2\right)}$$

In[32]:= `D1 = ωn^2 * Ωs * Ps / k / (2 * I * Ωs * ((I * Ωs - ζ * ωn)^2 + ωd^2))`

Out[32]= $$-\frac{i\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n + i\, \Omega_s)^2\right)}$$

In[33]:=

```
u[t_] = A1 * Exp[(ζ * ωn + I * ωd) * t] +
  B1 * Exp[(ζ * ωn - I * ωd) * t] + C1 * Exp[-I * Ωs * t] + D1 * Exp[I * Ωs * t]
```

Out[33]= $$\frac{i\, e^{-i\, t\, \Omega_s}\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n - i\, \Omega_s)^2\right)} - \frac{i\, e^{i\, t\, \Omega_s}\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n + i\, \Omega_s)^2\right)} + \frac{i\, e^{t\, (-i\, \omega_d + \zeta\, \omega_n)}\, P_s\, \omega_n^2\, \Omega_s}{2\, k\, \omega_d\, \left((-i\, \omega_d + \zeta\, \omega_n)^2 + \Omega_s^2\right)} - \frac{i\, e^{t\, (i\, \omega_d + \zeta\, \omega_n)}\, P_s\, \omega_n^2\, \Omega_s}{2\, k\, \omega_d\, \left((i\, \omega_d + \zeta\, \omega_n)^2 + \Omega_s^2\right)}$$

In[34]:= `ExpToTrig[%]`

Out[34]= $$\frac{i\, (\mathrm{Cos}[t\, \Omega_s] - i\, \mathrm{Sin}[t\, \Omega_s])\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n - i\, \Omega_s)^2\right)} - \frac{i\, (\mathrm{Cos}[t\, \Omega_s] + i\, \mathrm{Sin}[t\, \Omega_s])\, P_s\, \omega_n^2}{2\, k\, \left(\omega_d^2 + (-\zeta\, \omega_n + i\, \Omega_s)^2\right)} + \left(i\, (\mathrm{Cos}[t\, \omega_d + i\, t\, \zeta\, \omega_n] - i\, \mathrm{Sin}[t\, \omega_d + i\, t\, \zeta\, \omega_n])\, P_s\, \omega_n^2\, \Omega_s\right) \Big/ \left(2\, k\, \omega_d\, \left((-i\, \omega_d + \zeta\, \omega_n)^2 + \Omega_s^2\right)\right) - \left(i\, (\mathrm{Cos}[t\, \omega_d - i\, t\, \zeta\, \omega_n] + i\, \mathrm{Sin}[t\, \omega_d - i\, t\, \zeta\, \omega_n])\, P_s\, \omega_n^2\, \Omega_s\right) \Big/ \left(2\, k\, \omega_d\, \left((i\, \omega_d + \zeta\, \omega_n)^2 + \Omega_s^2\right)\right)$$

In[35]:= `FullSimplify[%]`

Out[35]= $$-\frac{1}{2\, k}\, i\, P_s\, \omega_n^2 \left(\frac{e^{i\, t\, \Omega_s}}{\omega_d^2 + (\zeta\, \omega_n - i\, \Omega_s)^2} - \frac{e^{-i\, t\, \Omega_s}}{\omega_d^2 + (\zeta\, \omega_n + i\, \Omega_s)^2} + \frac{1}{\omega_d}\, e^{-i\, t\, \omega_d + t\, \zeta\, \omega_n}\, \Omega_s \left(\frac{1}{(\omega_d + i\, \zeta\, \omega_n)^2 - \Omega_s^2} + \frac{e^{2\, i\, t\, \omega_d}}{(i\, \omega_d + \zeta\, \omega_n)^2 + \Omega_s^2}\right)\right)$$

In[50]:= `(*m = 50;*)`

In[36]:= `(*ζ = 0.1;*)`

In[37]:= `(*ωn = Sqrt[k/m];*)`

In[38]:= `(*k = 10;*)`

In[39]:= `(*Ps = 10;*)`

In[40]:= `(*ωd= ωn*Sqrt[1 - ζ^2];*)`

In[41]:= `(*Ωs = 1;*)`

In[42]:= `(*c= 2*ζ*ωn*m;*)`

In[43]:= **ExpToTrig[u[t]] // Simplify**

Out[43]=

$$\frac{1}{2\,k}\,\mathrm{i}\,P_s\,\omega_n^2\left(-\frac{\mathrm{Cos}[t\,\Omega_s]+\mathrm{i}\,\mathrm{Sin}[t\,\Omega_s]}{\omega_d^2+(\zeta\,\omega_n-\mathrm{i}\,\Omega_s)^2}+\frac{\mathrm{Cos}[t\,\Omega_s]-\mathrm{i}\,\mathrm{Sin}[t\,\Omega_s]}{\omega_d^2+(\zeta\,\omega_n+\mathrm{i}\,\Omega_s)^2}+\left((\mathrm{Cos}[t\,(\omega_d+\mathrm{i}\,\zeta\,\omega_n)]-\mathrm{i}\,\mathrm{Sin}[t\,(\omega_d+\mathrm{i}\,\zeta\,\omega_n)])\,\Omega_s\right)\Big/\left(\omega_d\left((-\mathrm{i}\,\omega_d+\zeta\,\omega_n)^2+\Omega_s^2\right)\right)-\left((\mathrm{Cos}[t\,(\omega_d-\mathrm{i}\,\zeta\,\omega_n)]+\mathrm{i}\,\mathrm{Sin}[t\,(\omega_d-\mathrm{i}\,\zeta\,\omega_n)])\,\Omega_s\right)\Big/\left(\omega_d\left((\mathrm{i}\,\omega_d+\zeta\,\omega_n)^2+\Omega_s^2\right)\right)\right)$$

In[44]:= **X = P_s / k / Sqrt[(1 - (Ω_s / ω_n) ^2) ^2 + (2 * ζ * Ω_s / ω_n) ^2] ;**

In[45]:= **α = ArcTan[-c * Ω_s / (k - m * Ω_s ^2)];**

In[46]:= **Bb = -X * Sin[-α];**

In[47]:= **Ab = ζ * ω_n * Bb / ω_d - Ω_s / ω_d * X * Cos[-α];**

In[48]:= **u2[t_] = Exp[-ζ * ω_n * t] * (Ab * Sin[ω_d * t] + Bb * Cos[ω_d * t]) + X * Sin[Ω_s * t - α];**

In[49]:= **(*Plot[{u[t],u2[t]},{t,0,10}]*)**

Problem 1 Part D

Defining ode

In[36]:=
```
ClearAll["Global`*"]
```

In[37]:=
```
de = m*u''[t] + c*u'[t] + k*u[t] == P_co Cos[Ω_co*t]
```

Out[37]= $k\,u[t] + c\,u'[t] + m\,u''[t] == \mathrm{Cos}[t\,\Omega_{co}]\,P_{co}$

Taking the Laplace transform

In[38]:=
```
lt = LaplaceTransform[de, t, s]
```

Out[38]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,(s\,\mathrm{LaplaceTransform}[u[t], t, s] - u[0]) + m\left(s^2\,\mathrm{LaplaceTransform}[u[t], t, s] - s\,u[0] - u'[0]\right) == \frac{s\,P_{co}}{s^2 + \Omega_{co}^2}$

Applying initial conditions

In[39]:=
```
U1 = lt /. u[0] → 0
```

Out[39]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,s\,\mathrm{LaplaceTransform}[u[t], t, s] + m\left(s^2\,\mathrm{LaplaceTransform}[u[t], t, s] - u'[0]\right) == \frac{s\,P_{co}}{s^2 + \Omega_{co}^2}$

In[40]:=
```
U2 = U1 /. u'[0] → 0
```

Out[40]= $k\,\mathrm{LaplaceTransform}[u[t], t, s] + c\,s\,\mathrm{LaplaceTransform}[u[t], t, s] + m\,s^2\,\mathrm{LaplaceTransform}[u[t], t, s] == \frac{s\,P_{co}}{s^2 + \Omega_{co}^2}$

Solving for U(s)

In[41]:=
```
U3 = Solve[U2, LaplaceTransform[u[t], t, s]] // Flatten
```

Out[41]= $\left\{\mathrm{LaplaceTransform}[u[t], t, s] \to \frac{s\,P_{co}}{\left(k + c\,s + m\,s^2\right)\left(s^2 + \Omega_{co}^2\right)}\right\}$

Inserting relations for m, ζ, ω_n and ω_d

In[42]:=
```
U4 = U3 /. c → 2*ζ*ω_n*m
```

Out[42]= $\left\{\mathrm{LaplaceTransform}[u[t], t, s] \to \frac{s\,P_{co}}{\left(k + m\,s^2 + 2\,m\,s\,\zeta\,\omega_n\right)\left(s^2 + \Omega_{co}^2\right)}\right\}$

In[43]:= `U5 = U4 /. m → k / ω_n^2`

Out[43]= $\left\{\text{LaplaceTransform}[u[t], t, s] \to \frac{s\, P_{co}}{\left(k + \frac{k s^2}{\omega_n^2} + \frac{2 k s \zeta}{\omega_n}\right)\left(s^2 + \Omega_{co}^2\right)}\right\}$

Expanding

In[44]:= `U6 = Apart[U5]`

Out[44]= $\left\{\text{LaplaceTransform}[u[t], t, s] \to \frac{s\, P_{co}\, \omega_n^2}{k\left(s^2 + 2 s \zeta \omega_n + \omega_n^2\right)\left(s^2 + \Omega_{co}^2\right)}\right\}$

In[51]:= `U7 = ApartSquareFree[U6, s]`

Out[51]= $\Big\{\text{LaplaceTransform}[u[t], t, s] \to \left(-s P_{co} \omega_n^4 - 2 \zeta P_{co} \omega_n^5 + s P_{co} \omega_n^2 \Omega_{co}^2\right) \Big/ \left(k\left(s^2 + 2 s \zeta \omega_n + \omega_n^2\right)\left(\omega_n^4 - 2 \omega_n^2 \Omega_{co}^2 + 4 \zeta^2 \omega_n^2 \Omega_{co}^2 + \Omega_{co}^4\right)\right) + \frac{s P_{co} \omega_n^4 - s P_{co} \omega_n^2 \Omega_{co}^2 + 2 \zeta P_{co} \omega_n^3 \Omega_{co}^2}{k\left(s^2 + \Omega_{co}^2\right)\left(\omega_n^4 - 2 \omega_n^2 \Omega_{co}^2 + 4 \zeta^2 \omega_n^2 \Omega_{co}^2 + \Omega_{co}^4\right)}\Big\}$

Taking inverse Laplace transform

In[53]:= `u1 = InverseLaplaceTransform[U7, s, t] // FullSimplify`

Out[53]= $\Big\{u[t] \to \Big(e^{-t\left(\zeta \omega_n + \sqrt{(-1+\zeta^2)\omega_n^2}\right)} P_{co}\, \omega_n \Big(\zeta \omega_n^2 \sqrt{(-1+\zeta^2)\omega_n^2} - e^{2 t \sqrt{(-1+\zeta^2)\omega_n^2}} \zeta \omega_n^2 \sqrt{(-1+\zeta^2)\omega_n^2} + \zeta \sqrt{(-1+\zeta^2)\omega_n^2}\, \Omega_{co}^2 - e^{2 t \sqrt{(-1+\zeta^2)\omega_n^2}} \zeta \sqrt{(-1+\zeta^2)\omega_n^2}\, \Omega_{co}^2 - \left(1 + e^{2 t \sqrt{(-1+\zeta^2)\omega_n^2}}\right)\left(-1+\zeta^2\right)\omega_n\left(\omega_n^2 - \Omega_{co}^2\right) + 2 e^{t\left(\zeta \omega_n + \sqrt{(-1+\zeta^2)\omega_n^2}\right)}\left(-1+\zeta^2\right)\omega_n\left(2 \zeta \,\text{Sin}[t\, \Omega_{co}]\, \omega_n \Omega_{co} + \text{Cos}[t\, \Omega_{co}]\left(\omega_n^2 - \Omega_{co}^2\right)\right)\Big)\Big) \Big/ \left(2 k\left(-1+\zeta^2\right)\left(\omega_n^4 + 2\left(-1 + 2\zeta^2\right)\omega_n^2 \Omega_{co}^2 + \Omega_{co}^4\right)\right)\Big\}$

Using assumptions 0 < ζ < 1 and ω_n > 0

In[55]:= `u2 = Refine[u1, ζ > 0 && ζ < 1 && ω_n > 0] // FullSimplify`

Out[55]= $\Big\{u[t] \to \Big(e^{-t\left(\zeta + i\sqrt{1-\zeta^2}\right)\omega_n} P_{co}\, \omega_n^2 \Big(i \zeta \sqrt{1-\zeta^2}\, \omega_n^2 - i\, e^{2 i t \sqrt{1-\zeta^2}\, \omega_n} \zeta \sqrt{1-\zeta^2}\, \omega_n^2 + i \zeta \sqrt{1-\zeta^2}\, \Omega_{co}^2 - i\, e^{2 i t \sqrt{1-\zeta^2}\, \omega_n} \zeta \sqrt{1-\zeta^2}\, \Omega_{co}^2 - \left(1 + e^{2 i t \sqrt{1-\zeta^2}\, \omega_n}\right)\left(-1+\zeta^2\right)\left(\omega_n^2 - \Omega_{co}^2\right) + 2 e^{t\left(\zeta + i\sqrt{1-\zeta^2}\right)\omega_n}\left(-1+\zeta^2\right)\left(2 \zeta \,\text{Sin}[t\, \Omega_{co}]\, \omega_n \Omega_{co} + \text{Cos}[t\, \Omega_{co}]\left(\omega_n^2 - \Omega_{co}^2\right)\right)\Big)\Big) \Big/ \left(2 k\left(-1+\zeta^2\right)\left(\omega_n^4 + 2\left(-1 + 2\zeta^2\right)\omega_n^2 \Omega_{co}^2 + \Omega_{co}^4\right)\right)\Big\}$

In[56]:= **u3 = u2 /. Sqrt[1 - ζ^2] * ω_n → ω_d // FullSimplify**

Out[56]=
$$\left\{u[t] \to \left(e^{-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n} P_{co}\,\omega_n^2 \left(i\,\zeta\sqrt{1-\zeta^2}\,\omega_n^2 - i\,e^{2\,i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\omega_n^2 + i\,\zeta\sqrt{1-\zeta^2}\,\Omega_{co}^2 - i\,e^{2\,i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\Omega_{co}^2 - \left(1+e^{2\,i\,t\,\omega_d}\right)\left(-1+\zeta^2\right)\left(\omega_n^2-\Omega_{co}^2\right) + 2\,e^{t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}\left(-1+\zeta^2\right)\left(2\,\zeta\,\mathrm{Sin}[t\,\Omega_{co}]\,\omega_n\,\Omega_{co} + \mathrm{Cos}[t\,\Omega_{co}]\left(\omega_n^2-\Omega_{co}^2\right)\right)\right)\right) \Big/ \left(2\,k\left(-1+\zeta^2\right)\left(\omega_n^4 + 2\left(-1+2\,\zeta^2\right)\omega_n^2\,\Omega_{co}^2 + \Omega_{co}^4\right)\right)\right\}$$

In[59]:= **u4 = ExpToTrig[u3] // FullSimplify**

Out[59]=
$$\left\{u[t] \to \left(e^{-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n} P_{co}\,\omega_n^2 \left(e^{i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,\omega_n^2 + e^{i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,\Omega_{co}^2 - \frac{1}{2}\left(1+e^{2\,i\,t\,\omega_d}\right)\left(-1+\zeta^2\right)\left(\omega_n^2-\Omega_{co}^2\right) + e^{t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}\left(-1+\zeta^2\right)\left(2\,\zeta\,\mathrm{Sin}[t\,\Omega_{co}]\,\omega_n\,\Omega_{co} + \mathrm{Cos}[t\,\Omega_{co}]\left(\omega_n^2-\Omega_{co}^2\right)\right)\right)\right) \Big/ \left(k\left(-1+\zeta^2\right)\left(\omega_n^4 + 2\left(-1+2\,\zeta^2\right)\omega_n^2\,\Omega_{co}^2 + \Omega_{co}^4\right)\right)\right\}$$

In[60]:= **u5 = u4 /. Sqrt[1 - ζ^2] * ω_n → ω_d**

Out[60]=
$$\left\{u[t] \to \left(e^{-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n} P_{co}\,\omega_n^2 \left(e^{i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,\omega_n^2 + e^{i\,t\,\omega_d}\,\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,\Omega_{co}^2 - \frac{1}{2}\left(1+e^{2\,i\,t\,\omega_d}\right)\left(-1+\zeta^2\right)\left(\omega_n^2-\Omega_{co}^2\right) + e^{t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}\left(-1+\zeta^2\right)\left(2\,\zeta\,\mathrm{Sin}[t\,\Omega_{co}]\,\omega_n\,\Omega_{co} + \mathrm{Cos}[t\,\Omega_{co}]\left(\omega_n^2-\Omega_{co}^2\right)\right)\right)\right) \Big/ \left(k\left(-1+\zeta^2\right)\left(\omega_n^4 + 2\left(-1+2\,\zeta^2\right)\omega_n^2\,\Omega_{co}^2 + \Omega_{co}^4\right)\right)\right\}$$

In[61]:= **Apart[u5]**

Out[61]=
$$\left\{u[t]\to -\frac{e^{-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}P_{co}\,\omega_n^4}{2\,k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}+\frac{\mathrm{Cos}[t\,\Omega_{co}]\,P_{co}\,\omega_n^4}{k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}+\right.$$
$$\frac{e^{i\,t\,\omega_d-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,P_{co}\,\omega_n^4}{k\left(-1+\zeta^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}+\frac{2\,\zeta\,\mathrm{Sin}[t\,\Omega_{co}]\,P_{co}\,\omega_n^3\,\Omega_{co}}{k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}+$$
$$\frac{e^{-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}P_{co}\,\omega_n^2\,\Omega_{co}^2}{2\,k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}-\frac{\mathrm{Cos}[t\,\Omega_{co}]\,P_{co}\,\omega_n^2\,\Omega_{co}^2}{k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}+$$
$$\left.\frac{e^{i\,t\,\omega_d-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}\zeta\sqrt{1-\zeta^2}\,\mathrm{Sin}[t\,\omega_d]\,P_{co}\,\omega_n^2\,\Omega_{co}^2}{k\left(-1+\zeta^2\right)\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}-\frac{e^{2\,i\,t\,\omega_d-t\left(\zeta+i\sqrt{1-\zeta^2}\right)\omega_n}P_{co}\,\omega_n^2\left(\omega_n^2-\Omega_{co}^2\right)}{2\,k\left(\omega_n^4-2\,\omega_n^2\,\Omega_{co}^2+4\,\zeta^2\,\omega_n^2\,\Omega_{co}^2+\Omega_{co}^4\right)}\right\}$$

Problem 3

Deriving equation for response of undamped sdof system to sinusoidal loading

EOM

```
de = m*u''[t] + k*u[t] == A*Sin[Ω*t]
```

$k\,u[t] + m\,u''[t] == A\,\text{Sin}[t\,\Omega]$

Solution

```
solution = DSolve[{de, u[0] == 0, u'[0] == 0}, u[t], t] // Flatten // FullSimplify
```

$$\left\{u[t] \to \frac{A\left(-\frac{\sqrt{m}\ \Omega\,\text{Sin}\left[\frac{\sqrt{k}\ t}{\sqrt{m}}\right]}{\sqrt{k}} + \text{Sin}[t\,\Omega]\right)}{k - m\,\Omega^2}\right\}$$

Simplifying

```
sol2 = solution /. m → k / ωn^2 // FullSimplify
```

$$\left\{u[t] \to \frac{A\,\omega_n\,(-\Omega\,\text{Sin}[t\,\omega_n] + \text{Sin}[t\,\Omega]\,\omega_n)}{k\left(-\Omega^2 + \omega_n^2\right)}\right\}$$

Integrating to find b_n

Inputting terms to integrate

```
f1 := 4*P0 / T0 * t;

f2 := (-4*P0 / T0 * t + 2*P0);

f3 := 4*P0 / T0 * t - 4*P0;

sTerm := Sin[2*n*Pi*t / T0];
```

Integrating

```
b = Integrate[f1 * sTerm, {t, 0, T0 / 4}] + Integrate[f2 * sTerm, {t, T0 / 4, 3 * T0 / 4}] +
   Integrate[f3 * sTerm, {t, 3 * T0 / 4, T0}] // FullSimplify
```

$$\frac{\left(2\,\text{Sin}\left[\frac{n\,\pi}{2}\right] - 2\,\text{Sin}\left[\frac{3\,n\,\pi}{2}\right] + \text{Sin}[2\,n\,\pi]\right) P_0\, T_0}{n^2\,\pi^2}$$

$$\sum_{n=1}^{\infty} \frac{\left(2\,\text{Sin}\left[\frac{n\,\pi}{2}\right] - 2\,\text{Sin}\left[\frac{3\,n\,\pi}{2}\right] + \text{Sin}[2\,n\,\pi]\right) P_0\, T_0}{n^2\,\pi^2} \,/$$

$$\frac{4\,\text{Catalan}\, P_0\, T_0}{\pi^2}$$

Problem 4

First interval

```
(*m = 2;*)

(*k = 32;*)

(*T0 = 2;*)

(*T1= T0/4;*)

(*P0=64;*)

(*ωn= Sqrt[k/m];*)

u1 = Integrate[P0 / (m * ωn * T1) * τ * Sin[ωn * (t - τ)], {τ, 0, t}] // FullSimplify
```

$$\frac{P_0\,(-\text{Sin}[t\,\omega_n] + t\,\omega_n)}{m\,T_1\,\omega_n^3}$$

```
(*Plot[u1,{t,0,T1}]*)
```

Second interval

```
u2 = Integrate[P0 / (m * ωn * T1) * τ * Sin[ωn * (t - τ)], {τ, 0, T1}] + Integrate[
    1 / (m * ωn) * (-P0 / T1 * τ + 2 * P0) * Sin[ωn * (t - τ)], {τ, T1, t}] // FullSimplify
```

$$-\frac{P_0\,(\text{Sin}[t\,\omega_n] - 2\,\text{Sin}[(t - T_1)\,\omega_n] + (t - 2\,T_1)\,\omega_n)}{m\,T_1\,\omega_n^3}$$

```
(*Plot[Piecewise[{{u1,t<T1},{u2,t>T1}}],{t,0,3*T1}]*)
```

Third interval

```
Integrate[P0 / (m * ωn * T1) * τ * Sin[ωn * (t - τ)], {τ, 0, T1}] +
  Integrate[1 / (m * ωn) * (-P0 / T1 * τ + 2 * P0) * Sin[ωn * (t - τ)], {τ, T1, 3 * T1}] +
  Integrate[1 / (m * ωn) * (P0 / T1 * τ - 4 * P0) * Sin[ωn * (t - τ)],
   {τ, 3 * T1, t}] // FullSimplify
```

$$-\frac{P_0\,(\text{Sin}[t\,\omega_n] + 2\,\text{Sin}[(t - 3\,T_1)\,\omega_n] - 2\,\text{Sin}[(t - T_1)\,\omega_n] - (t - 4\,T_1)\,\omega_n)}{m\,T_1\,\omega_n^3}$$

Problem 7

Inputs

In[219]:=
```
ClearAll["Global`*"]
```

In[220]:=
```
m = 40.;
```

In[221]:=
```
L = 10.;
```

In[222]:=
```
k = 10 000.;
```

In[223]:=
```
k1 = 2 * k;
```

In[224]:=
```
k2 = k;
```

In[225]:=
```
a = L / 2;
```

In[226]:=
```
b = a;
```

In[227]:=
```
I0 = 1 / 12 * m * L^2;
```

Stiffness and Mass matrices

In[228]:=
```
M = {{m, L * m / 2}, {L * m / 2, L^2 * m / 3}}
```

Out[228]=
```
{{40., 200.}, {200., 1333.33}}
```

In[229]:=
```
K = {{k1 + k2, L * k2}, {L * k2, L^2 * k2}}
```

Out[229]= $\{\{30\,000., 100\,000.\}, \{100\,000., 1. \times 10^{6}\}\}$

Finding eigenvectors and eigenvalues

In[230]:=
```
Λ = Eigenvalues[Inverse[M].K]
```

Out[230]=
```
{2366.03, 633.975}
```

In[231]:=
```
Φ = Eigenvectors[Inverse[M].K]
```

Out[231]=
```
{{0.985329, -0.170664}, {0.985329, 0.170664}}
```

Separating Eigenvalues and Eigenvectors

In[232]:= $\lambda_1 = \Lambda[[2]]$

Out[232]= 633.975

In[233]:= $\lambda_2 = \Lambda[[1]]$

Out[233]= 2366.03

In[234]:= $\phi_1 = \Phi[[2, All]]$

Out[234]= {0.985329, 0.170664}

In[235]:= $\phi_2 = \Phi[[1, All]]$

Out[235]= {0.985329, -0.170664}

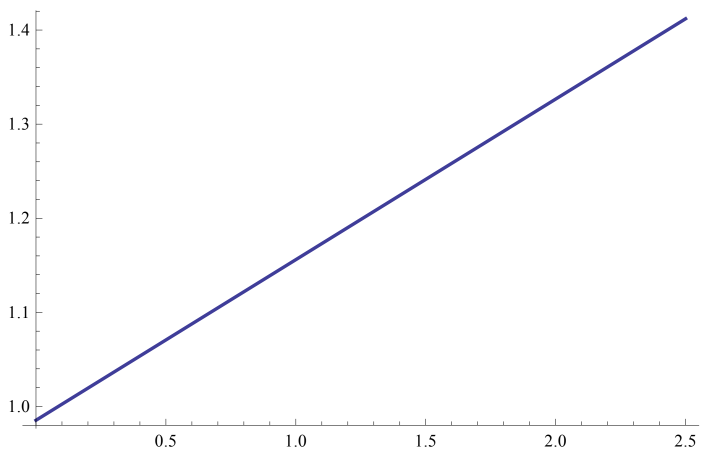
In[236]:= $F_x = \{x^0, x\}$

Out[236]= {1, x}

In[237]:= $\psi_1 = F_x . \phi_1$

Out[237]= 0.985329 + 0.170664 x

In[244]:= Plot[ψ_1, {x, 0, .25 * L}, PlotStyle → Thickness[.005]]

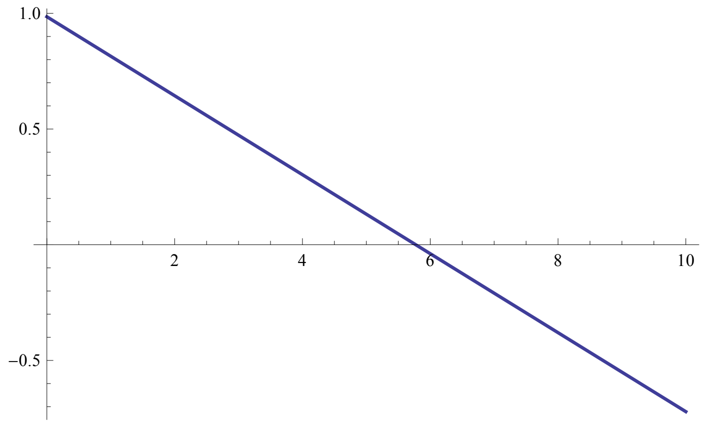
Out[244]=


In[239]:= $\psi_2 = F_x . \phi_2$

Out[239]= 0.985329 - 0.170664 x

In[245]:= `Plot[ψ2, {x, 0, L}, PlotStyle → Thickness[.005]]`

Out[245]=



Solving for node

In[241]:= `Xposition = Solve[ψ2 == 0, x] // Flatten`

Out[241]= `{x → 5.7735}`